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Quantum Stochastic Evolutions

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Quantum stochastic differential inclusions of hypermaximal monotone type are studied, under very general conditions, by means of certain discrete schemes which approximate them. The existence of an evolution operator corresponding to each such inclusion is proved.

1. INTRODUCTION

In Ekhaguere (1995), we introduced the class of quantum stochastic differential inclusions of *hypermaximal monotone* type and proved that, subject to some continuity condition on the resolvent of its associated hypermaximal monotone multifunction, every member of the class possesses a unique adapted solution. The class is interesting because its members model the temporal *evolution* of quantum systems. In this paper, we consider quantum stochastic differential inclusions involving multifunctions $\{P(t, \cdot): t \in [0, T], T > 0\}$ that are hypermaximal monotone for almost all $t \in [0, T]$ and study the problem of the existence of *evolution operators* corresponding to them. We work under assumptions that are substantially relaxed relative to those employed in Ekhaguere (1995) and adopt a constructive approach involving the solving of a discrete set of approximating inclusions. Ordinary differential inclusions of evolution type in arbitrary Banach spaces have been extensively studied in recent years by a number of authors (Crandall and Pazy, 1972; Crandall, 1973; Crandall and Evans, 1975; Kobayashi, 1975; Evans, 1977; Kobayashi *et al.*, 1984; Iwamiya *et al.*, 1986; Oharu, 1986). In this paper, we adapt the techniques and arguments of Evans (1977) in the Banach space context to the present noncommutative setting involving inclusions in certain locally convex spaces.

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A summary of the rest of the paper is as follows. In Section 2, we assemble some basic notions and results which are used throughout the paper. Section 3 describes the initial value quantum stochastic differential inclusion $(3.2.1)_X$ that is studied in the subsequent sections. As shown in Ekhaguere (1992), Problem $(3.2.1)_X$ is equivalent to Problem $(3.2.1)_P$. The hypermaximal monotone multifunctions associated with these inclusions are time-dependent. We prove that, under the hypotheses introduced in this section, the generalized domains (Crandall, 1973) of these multifunctions are essentially independent of time. Using a nice choice of partitions of the time interval, we introduce discrete schemes which approximate Problem $(3.2.1)_P$ in Section 4. Proposition 4.1 gives information on how the solutions of two such schemes compare. Section 5 contains the proofs of two fundamental results: Theorems 5.1 and 5.6. These assert the uniform convergence of the sequence of solutions of the approximating schemes. The limit solutions of two inclusions $(3.2.1)_X$ and $(3.2.1)_Y$ are compared in Section 6. In Section 7, we describe the *evolution operator* associated with a limit solution and interpret the multifunction P as its generator. The final Section 8 establishes the relationship between a limit solution of Problem $(3.2.1)_X$ and a solution of the same problem. It is shown in Theorem 8.1 that a solution of Problem $(3.2.1)_X$ may be realized as the limit of solutions of approximating schemes of Problem $(3.2.1)_P$. As a corollary, it is proved that if Problem $(3.2.1)_X$ has a solution, then it must coincide with the limit of the solutions of its approximating schemes.

2. BASIC NOTIONS AND RESULTS

The setting of this paper is essentially as in Ekhaguere (1992, 1995). We first assemble some of the important notions and results which are employed in the subsequent discussion.

Let $\mathcal{X}$ be a linear space and $n \in \mathbf{N}$, the natural numbers. Then $\mathcal{X}^n$ [resp. $\mathcal{X}^{(n)}$] denotes the *Cartesian product* [resp. *algebraic tensor product*] of n copies of $\mathcal{X}$. The n-fold Hilbert space tensor product (Reed and Simon, 1972) of a Hilbert space $\mathcal{X}$ with itself will also be denoted by $\mathcal{X}^{(n)}$. Given two subsets A, B of $\mathcal{X}$, a point $c \in \mathcal{X}$, and scalars α, $\beta \in \mathbf{C}$, the complex numbers, we define $\alpha A + \beta B$ by

$$\alpha A + \beta B = \{\alpha a + \beta b: a \in A, b \in B\}$$

and $c + A$ by

$$c + A = \{c + a: a \in A\}$$

We use the notation $\mathrm{sesq}(\mathcal{X})$ for the set of all *sesquilinear forms* on the linear space $\mathcal{X}$. A member p of $\mathrm{sesq}(\mathcal{X})$ will be assumed *conjugate-linear* on the left and its value at the point $(x, y) \in \mathcal{X}^2$ will be written as $p(x, y)$.

2.1. Fock Space Setting

To each pair (I, Y) consisting of a subinterval $I \subseteq \mathbb{R}_+ \equiv [0, \infty)$ and a complex Hilbert space Y, with $B(\mathrm{Y})$ as its Banach space of bounded endomorphisms, we associate the linear space $L^2_{\mathrm{Y}}(I)$ of Bochner square-integrable Y-valued functions on I and the linear space $L^\infty_{\mathrm{Y},\mathrm{loc}}(I)$ [resp. $L^\infty_{B(\mathrm{Y}),\mathrm{loc}}(I)$] of all measurable, locally bounded functions from I to Y [resp. to $B(\mathrm{Y})$]. The space $L^\infty_{\mathrm{Y},\mathrm{loc}}(I)$ is a left $L^\infty_{B(\mathrm{Y}),\mathrm{loc}}(I)$-module if, for $f \in L^\infty_{\mathrm{Y},\mathrm{loc}}(I)$ and $\pi \in L^\infty_{B(\mathrm{Y}),\mathrm{loc}}(I)$, the member $\pi f \in L^\infty_{\mathrm{Y},\mathrm{loc}}(I)$ is defined by $(\pi f)(t) = \pi(t)f(t)$, for almost all $t \in I$.

If $\mathbf{D}$ is a complex pre-Hilbert space and $\mathbf{H}$ its completion, we write $L^+_w(\mathbf{D}, \mathbf{H})$ for the linear space of all linear maps x from $\mathbf{D}$ to $\mathbf{H}$ such that the domain of the operator adjoint x^* of x contains $\mathbf{D}$, and $\Gamma(\mathbf{H})$ for the *boson Fock space* (Guichardet, 1972) over $\mathbf{H}$. For $f \in \mathbf{H}$, define $\otimes^0 f = 1$ and if $n \geq 1$, define $\otimes^n f$ as the n-fold tensor product of f with itself. Then

$$e(f) = \bigoplus_{n=0}^{\infty} (n!)^{1/2} \otimes^n f$$

lies in $\Gamma(\mathbf{H})$ and is the *exponential vector* associated with f. The exponential vectors in $\Gamma(\mathbf{H})$ generate a dense subspace.

Throughout the paper, $\mathbb{D}$ is a complex pre-Hilbert space with completion $\mathfrak{R}$, and $\mathbb{E}$, $\mathbb{E}_t$, and $\mathbb{E}^t$, $t > 0$, are the linear spaces generated by the exponential vectors in $\Gamma(L^2_{\mathrm{Y}}(\mathbb{R}_+))$, $\Gamma(L^2_{\mathrm{Y}}([0, t)))$, and $\Gamma(L^2_{\mathrm{Y}}([t, \infty)))$, $t > 0$, respectively. The *inner product* and *norm* of the Hilbert space $\mathfrak{R} \otimes \Gamma(L^2_{\mathrm{Y}}(\mathbb{R}_+))$ will be denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. We write $\mathcal{A}$, $\mathcal{A}_t$, and $\mathcal{A}^t$ for the linear spaces of linear operators defined as follows:

$$\mathcal{A} = L^+_w(\mathbb{D} \underline{\otimes} \mathbb{E}, \mathfrak{R} \otimes \Gamma(L^2_{\mathrm{Y}}(\mathbb{R}_+)))$$

$$\mathcal{A}_t = L^+_w(\mathbb{D} \underline{\otimes} \mathbb{E}_t, \mathfrak{R} \otimes \Gamma(L^2_{\mathrm{Y}}([0, t)))) \otimes 1^t$$

$$\mathcal{A}^t = 1_t \otimes L^+_w(\mathbb{E}^t, \Gamma(L^2_{\mathrm{Y}}([t, \infty)))), \qquad t > 0$$

where $\underline{\otimes}$ denotes *algebraic tensor product* throughout the paper and 1_t [resp. 1^t] is the identity map on $\mathfrak{R} \otimes \Gamma(L^2_{\mathrm{Y}}([0, t)))$ [resp. $\Gamma(L^2_{\mathrm{Y}}([t, \infty)))$], $t > 0$. It is evident that $\mathcal{A}_t$ and $\mathcal{A}^t$, $t > 0$, are linear subspaces of $\mathcal{A}$. This space will be endowed with the locally convex topology τ_w whose generating family $\{\|\cdot\|_{\eta,\xi}: \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$ of seminorms is defined by

$$\|x\|_{\eta,\xi} = |\langle \eta, x\xi \rangle|, \qquad x \in \mathcal{A}, \quad \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$$

The completions of the locally convex spaces $(\mathcal{A}, \tau_w)$, $(\mathcal{A}_t, \tau_w)$, and $(\mathcal{A}^t, \tau_w)$, $t > 0$, will be denoted by $\tilde{\mathcal{A}}$, $\tilde{\mathcal{A}}_t$, and $\tilde{\mathcal{A}}^t$, $t > 0$, respectively.

The net $\{\tilde{\mathcal{A}}_t: t \in \mathbb{R}_+\}$ furnishes a filtration of $\tilde{\mathcal{A}}$.

In the sequel, we denote $\Gamma(L^2_\gamma(\mathbb{R}_+))$ simply by Γ. We write $\mathbb{1}$ for the identity map on $\mathfrak{R} \otimes \Gamma$ and $\langle\cdot, \cdot\rangle_{(2)}$ for the inner product of the Hilbert space $(\mathfrak{R} \otimes \Gamma)^{(2)}$.

2.2. Tangent Functionals

For $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ and $0 \neq \lambda \in \mathbb{R}$, introduce the $\mathbb{R}$-valued functionals $[\cdot, \cdot]_{\eta\xi}$ and $[\cdot, \cdot]_{\lambda,\eta\xi}$ on $\tilde{\mathcal{A}} \times \tilde{\mathcal{A}}$ by

$$[u, v]_{\eta\xi} = \frac{\mathrm{Re}(\langle u\xi, \eta\rangle\langle\eta, v\xi\rangle)}{\|v\|_{\eta,\xi}}$$

and

$$[u, v]_{\lambda,\eta\xi} = \frac{\|v + \lambda u\|_{\eta,\xi} - \|v\|_{\eta,\xi}}{\lambda}$$

$u, v \in \tilde{\mathcal{A}}$, where $\mathrm{Re}(\cdots)$ always denotes the *real part* of $(\cdots)$. One verifies that

$$|[u, v]_{\lambda,\eta\xi} - [y, z]_{\lambda,\eta\xi}\| \leq \frac{2}{\lambda}\|u - y\|_{\eta,\xi} + \|v - z\|_{\eta,\xi}$$

Moreover, the family $\{[\cdot, \cdot]_{\eta\xi}: \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$ enjoys the following easily checked properties.

Proposition 2.1. For $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, $u, v, y, z \in \tilde{\mathcal{A}}$, $\alpha \in \mathbb{C}$, and $s, t \in \mathbb{R}$:
(i) $[u, v]_{\eta\xi} = \lim_{\lambda\to 0}[u, v]_{\lambda,\eta\xi}$.
(ii) $[\alpha u + z, u]_{\eta\xi} = \mathrm{Re}(\alpha)\,\|u\|_{\eta,\xi} + [z, u]_{\eta\xi}$.
(iii) $[su, tv]_{\eta\xi} = (st/|t|)[u, v]_{\eta\xi}$.
(iv) $[u + z, v]_{\eta\xi} = [u, v]_{\eta\xi} + [z, v]_{\eta\xi}$.
(v) $[u, v]_{\eta\xi} \leq \|u + v\|_{\eta,\xi} - \|v\|_{\eta,\xi}$.
(vi) $|[u, v]_{\eta\xi}| \leq \|u\|_{\eta,\xi}$.

We shall employ these results in the sequel.

2.3. Multifunctions

Much of the subsequent analysis will focus largely on *multifunctions* (also called *set-valued* or *multivalued maps*) (Aubin and Cellina, 1984; Kisielewicz, 1991) from $\tilde{\mathcal{A}}$ to $2^{\mathcal{S}}$, where $\mathcal{S} = \tilde{\mathcal{A}}$, sesq($\mathbb{D} \underline{\otimes} \mathbb{E}$), or sesq($(\mathbb{D} \underline{\otimes} \mathbb{E})^{(2)}$). If $\mathcal{P}: \tilde{\mathcal{A}} \to 2^{\mathcal{S}}$, then its *domain* is $D(\mathcal{P}) = \{x \in \tilde{\mathcal{A}}: \mathcal{P}(x) \neq \emptyset\}$, *range* is range$(\mathcal{P}) = \cup_{x\in\tilde{\mathcal{A}}} \mathcal{P}(x)$, and *graph* is graph$(\mathcal{P}) = \{(x, y) \in \tilde{\mathcal{A}} \times \mathcal{S}: y \in \mathcal{P}(x)\}$.

The sum of two multifunctions $\mathcal{P}$, $\mathcal{Q}$ from $\tilde{\mathcal{A}}$ to $2^{\mathcal{S}}$ is defined by

$$(\mathcal{P} + \mathcal{Q})(x) = \begin{cases} \mathcal{P}(x) + \mathcal{Q}(x) & \text{if} \quad D(\mathcal{P}) \cap D(\mathcal{Q}) \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

$x \in \tilde{\mathcal{A}}$. Given

$$x_0 \in \tilde{\mathcal{A}}, \qquad \mathcal{P}: \quad \tilde{\mathcal{A}} \to 2^{\text{sesq}((\mathbb{D}\underline{\otimes}\mathbb{E})^{(2)})}, \qquad P: \quad \tilde{\mathcal{A}} \to 2^{\text{sesq}(\mathbb{D}\underline{\otimes}\mathbb{E})}$$

$$B: \quad \tilde{\mathcal{A}} \to 2^{\tilde{\mathcal{A}}}, \qquad \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}, \qquad \zeta_1, \zeta_2 \in (\mathbb{D} \underline{\otimes} \mathbb{E})^{(2)}$$

we use the following notation:

$$\mathcal{P}(x)(\zeta_1, \zeta_2) = \{\mathbf{p}(\zeta_1, \zeta_2): \mathbf{p} \text{ is a sesquilinear form on } (\mathbb{D} \underline{\otimes} \mathbb{E})^{(2)} \text{ and } \mathbf{p} \in \mathcal{P}(x)\}, \quad x \in D(\mathcal{P})$$

$$P(x)(\eta, \xi) = \{p(\eta, \xi): p \text{ is a sesquilinear form on } \mathbb{D} \underline{\otimes} \mathbb{E} \text{ and } p \in P(x)\}, \qquad x \in D(P)$$

$$B(x)(\eta, \xi) = \{\langle \eta, b\xi \rangle: b \in B(x)\}, x \in D(B)$$

$$B(x) \otimes x_0 = \{b \otimes x_0: b \in B(x)\}, x \in D(B)$$

and denote the multifunction $x \mapsto B(x) \otimes x_0$ from $D(B) \subseteq \tilde{\mathcal{A}}$ to $2^{\tilde{\mathcal{A}} \times \tilde{\mathcal{A}}}$ by $B \otimes x_0$.

Definition. A multifunction $P: \tilde{\mathcal{A}} \to 2^{\text{sesq}(\mathbb{D}\underline{\otimes}\mathbb{E})}$ will be called *regular* if, for $\alpha, \beta \in L^{\infty}_{Y,\text{loc}}(\mathbb{R}_+)$, there is a multifunction $P_{\alpha\beta}: D(P) \subseteq \tilde{\mathcal{A}} \to 2^{\tilde{\mathcal{A}}}$ such that

$$P(x)(\eta, \xi) = \langle \eta, P_{\alpha\beta}(x)\xi \rangle, \qquad x \in D(P) \tag{2.3.1}$$

for all $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^{\infty}_{Y,\text{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$.

Remark. We will be interested in multifunctions

$$\mathcal{P}: \quad D(\mathcal{P}) \subseteq \tilde{\mathcal{A}} \to 2^{\text{sesq}((\mathbb{D}\underline{\otimes}\mathbb{E})^{(2)})}$$

which are associated with regular multifunctions from $\tilde{\mathcal{A}}$ to $2^{\text{sesq}(\mathbb{D}\underline{\otimes}\mathbb{E})}$ as follows.

Definition. We say that

$$\mathcal{P}: \quad D(\mathcal{P}) \subseteq \tilde{\mathcal{A}} \to 2^{\text{sesq}((\mathbb{D}\underline{\otimes}\mathbb{E})^{(2)})}$$

lies in the class $\text{Reg}(\mathcal{A})_0$ if there is a regular multifunction

$$P: \quad D(P) \subseteq \tilde{\mathcal{A}} \to 2^{\text{sesq}(\mathbb{D}\underline{\otimes}\mathbb{E})}$$

with representation as in (2.3.1), such that

$$\mathcal{P}(x)(\eta_1 \otimes \eta_2, \xi_1 \otimes \xi_2) = \langle \eta_1 \otimes \eta_2, (P_{\alpha_1\beta_1}(x) \otimes \mathbb{1})(\xi_1 \otimes \xi_2)\rangle_{(2)}, \qquad x \in D(P) \qquad (2.3.2)$$

for arbitrary $\eta_j, \xi_j \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta_j = u_j \otimes e(\alpha_j)$, $\xi_j = v_j \otimes e(\beta_j)$, $\alpha_j, \beta_j \in L^{\infty}_{\Upsilon,\mathrm{loc}}(\mathbb{R}_+)$, $u_j, v_j \in \mathbb{D}$, $j = 1, 2$.

Notation. When (2.3.2) holds, we shall often write $\mathcal{P} = P \otimes \mathbb{1}$.

2.4. Monotone Multifunctions

The notions of monotonicity for the members of the class $\mathrm{Reg}(\tilde{\mathcal{A}})_0$ which we employ in the sequel are described as follows [see Ekhaguere (1995) for a more general setting].

For $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, $\Phi_{(\eta,\xi)}(\cdot, \cdot)$ denotes the map from $\tilde{\mathcal{A}} \times \tilde{\mathcal{A}} \to \mathfrak{R} \otimes \Gamma$ given by

$$\Phi_{(\eta,\xi)}(x, y) = \eta \otimes (x - y)\xi, \qquad x, y \in \tilde{\mathcal{A}}$$

Notice that $\Phi_{(\eta,\xi)}(x, y) = \Phi_{(\eta,\xi)}(x - y, 0)$, $x, y \in \tilde{\mathcal{A}}$.

Let $(\eta, \xi) \in (\mathbb{D} \underline{\otimes} \mathbb{E})^2$. If $(\mathfrak{R} \otimes \Gamma)^{(2)}_{(\eta,\xi)}$ denotes the closure of $(\tilde{\mathcal{A}} \underline{\otimes} \tilde{\mathcal{A}})(\xi \otimes \eta)$ in $(\mathfrak{R} \otimes \Gamma)^{(2)}$, then $\Phi_{(\eta,\xi)}$ is a *global Φ-system* for the pair $(\tilde{\mathcal{A}}, (\mathfrak{R} \otimes \Gamma)^{(2)}_{(\eta,\xi)})$ over $\tilde{\mathcal{A}}$ in the sense of Browder (1976).

Definition. A member $\mathcal{P}$ of $\mathrm{Reg}(\tilde{\mathcal{A}})_0$, assumed already represented in the form (2.3.2), will be called:

(i) *Monotone* if

$$\mathrm{Re}(\langle (a - b)(\xi \otimes \eta), \Phi_{(\eta,\xi)}(x, y)\rangle_{(2)}) \geq 0$$

whenever $a \in P_{\alpha\beta}(x) \otimes \mathbb{1}$, $b \in P_{\alpha\beta}(y) \otimes \mathbb{1}$, $x, y \in D(\mathcal{P})$, and $\eta, \xi = \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^{\infty}_{\Upsilon,\mathrm{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$.

(ii) *Maximal monotone* if the graph of $\mathcal{P}$ is not properly contained in the graph of any other monotone member of $\mathrm{Reg}(\tilde{\mathcal{A}})_0$.

(iii) *Hypermaximal monotone* if $\mathcal{P}$ is monotone and (a) the range of the map

$$x \mapsto \mathrm{id}_{\tilde{\mathcal{A}}}(x) \otimes \mathbb{1} + P_{\alpha\beta}(x) \otimes \mathbb{1}, \qquad x \in D(\mathcal{P}), \quad \alpha, \beta \in L^{\infty}_{\Upsilon,\mathrm{loc}}(\mathbb{R}_+)$$

is all of $\tilde{\mathcal{A}} \otimes \mathbb{1}$, and (b) $(\mathrm{id}_{\tilde{\mathcal{A}}}(\cdot) + P_{\alpha\beta}(\cdot) \otimes \mathbb{1})^{-1}$, $\alpha, \beta \in L^{\infty}_{\Upsilon,\mathrm{loc}}(\mathbb{R}_+)$, is a continuous single-valued map from $\tilde{\mathcal{A}} \otimes \mathbb{1}$ to $D(\mathcal{P})$.

Here, $\mathrm{id}_{\tilde{\mathcal{A}}}(\cdot)$ is the *identity map* on $\tilde{\mathcal{A}}$.

Remark. 1. A number of results about monotone maps were established in Ekhaguere (1995); see, however, the Appendix to this paper.

2. Notice that $\mathcal{P} \in \text{Reg}(\mathcal{A})_0$, with representation of the form (2.3.2), is monotone iff $[y_1 - y_2, x_1 - x_2]_{\eta\xi} \geq 0$, for all pairs (x_1, y_1), (x_2, y_2) in the graph of $P_{\alpha\beta}$, for all η, $\xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, α, $\beta \in L^{\infty}_{\gamma,\text{loc}}(\mathbb{R}_+)$, u, $v \in \mathbb{D}$.

Notation. We denote by Hypmax$(I \times \tilde{\mathcal{A}})$ the set of all multifunctions $\mathcal{P}$, with domains in $I \times \tilde{\mathcal{A}}$ and values in $2^{\text{sesq}((\mathbb{D}\underline{\otimes}\mathbb{E})^{(2)})}$, such that for almost every $t \in I$, the multifunction

$$\mathcal{P}(t, \cdot): \quad D(\mathcal{P}(t, \cdot) \subseteq \tilde{\mathcal{A}} \to 2^{\text{sesq}((\mathbb{D}\underline{\otimes}\mathbb{E})^{(2)})}$$

is regular and hypermaximal monotone.

2.5. Resolvent and Yosida Approximation

Let $\mathcal{P} \in$ Hypmax$(I \times \tilde{\mathcal{A}})$. Then, for almost every $t \in I$, there is a regular multifunction

$$P(t, \cdot): \quad D(P(t, \cdot)) \subseteq \tilde{\mathcal{A}} \to 2^{\text{sesq}(\mathbb{D}\underline{\otimes}\mathbb{E})}$$

with the representation

$$P(t, x)(\eta, \xi) = \langle \eta, P_{\alpha\beta}(t, x)\xi \rangle, \qquad x \in D(P(t, \cdot))$$

for all η, $\xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, α, $\beta \in L^{\infty}_{\gamma,\text{loc}}(\mathbb{R}_+)$, u, $v \in \mathbb{D}$, such that

$$\begin{aligned} &\mathcal{P}(t, x)(\eta_1 \otimes \eta_2, \xi_1 \otimes \xi_2) \\ &\quad = \langle \eta_1 \otimes \eta_2, (P_{\alpha_1\beta_1}(t, x) \otimes 1)(\xi_1 \otimes \xi_2) \rangle_{(2)}, \qquad x \in D(P) \subseteq \tilde{\mathcal{A}} \end{aligned}$$

for arbitrary η_j, $\xi_j \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta_j = u_j \otimes e(\alpha_j)$, $\xi_j = v_j \otimes e(\beta_j)$, α_j, $\beta_j \in L^{\infty}_{\gamma,\text{loc}}(\mathbb{R}_+)$, u_j, $v_j \in \mathbb{D}$, $j = 1, 2$.

Let $\lambda > 0$ and α, $\beta \in L^{\infty}_{\gamma,\text{loc}}(\mathbb{R}_+)$. For almost all $t \in I$, define

$$J_{\lambda,\alpha\beta}(t, \cdot) = (\text{id}_{\tilde{\mathcal{A}}}(\cdot) + \lambda P_{\alpha\beta}(t, \cdot))^{-1}$$

$$P_{\lambda,\alpha\beta}(t, \cdot) = \frac{1}{\lambda}(\text{id}_{\tilde{\mathcal{A}}}(\cdot) - J_{\lambda,\alpha\beta}(t, \cdot))$$

These single-valued maps give rise to the sesquilinear forms $J_\lambda(t, x)$ and $P_\lambda(t, x)$, $(t, x) \in I \times \tilde{\mathcal{A}}$, defined by

$$J_\lambda(t, x)(\eta, \xi) = \langle \eta, J_{\lambda,\alpha\beta}(t, x)\xi \rangle$$

$$P_\lambda(t, x)(\eta, \xi) = \langle \eta, P_{\lambda,\alpha\beta}(t, x)\xi \rangle$$

for arbitrary η, $\xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, α, $\beta \in L^{\infty}_{\gamma,\text{loc}}(\mathbb{R}_+)$, u, $v \in \mathbb{D}$.

The single-valued maps $J_\lambda(t, \cdot)$ and $P_\lambda(t, \cdot)$ are called the *resolvent* and *Yosida approximation*, respectively, of the multifunction $P(t, \cdot)$.

We refer to Ekhaguere (1995) for a description of some of the properties of these maps. The following results will also be needed in the sequel.

Proposition 2.2. Let $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^\infty_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$. Then, the following statements are true.

(i) For $\lambda, \mu > 0$, $x \in \tilde{\mathcal{A}}$, and almost all $t \in I$,

$$J_{\lambda,\alpha\beta}(t, x) = J_{\mu,\alpha\beta}\left(t, \frac{\mu}{\lambda} x + \frac{\lambda - \mu}{\lambda} J_{\lambda,\alpha\beta}(t, x)\right)$$

(ii) For $0 < \mu \leq \lambda$, $(t, x) \in I \times \tilde{\mathcal{A}}$, and arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^\infty_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$,

$$|P_\lambda(t, x)(\eta, \xi)| \leq |P_\mu(t, x)(\eta, \xi)|$$

Proof. If $x \in \tilde{\mathcal{A}}$, then there is a point (x_0, y_0) in the graph of $P_{\alpha\beta}(t, \cdot)$ such that $x_0 + \lambda y_0 = x$. Since

$$\frac{\mu}{\lambda} x + \frac{\lambda - \mu}{\lambda} J_{\lambda,\alpha\beta}(t, x) = \frac{\mu}{\lambda}(x_0 + \lambda y_0) + \frac{\lambda - \mu}{\lambda} x_0 = x_0 + \mu y_0$$

and the pair $(x_0 + \mu y_0, x_0)$ is in the graph of the single-valued map $J_{\mu,\alpha\beta}(t, \cdot)$, one gets

$$x_0 = J_{\lambda,\alpha\beta}(t, x) = J_{\mu,\alpha\beta}(t, x_0 + \mu y_0) = J_{\mu,\alpha\beta}\left(t, \frac{\mu}{\lambda} x + \frac{\lambda - \mu}{\lambda} J_{\lambda,\alpha\beta}(t, x)\right)$$

(ii) This is proved as follows. Let $0 < \mu \leq \lambda$. Then

$$\begin{aligned}
&|P_\lambda(t, x)(\eta, \xi)| \\
&= |\langle \eta, P_{\lambda,\alpha\beta}(t, x)\xi\rangle| \\
&= \frac{1}{\lambda} |\langle \eta, (x - J_{\lambda,\alpha\beta}(t, x))\xi\rangle| \\
&\leq \frac{1}{\lambda} \|x - J_{\mu,\alpha\beta}(t, x)\|_{\eta,\xi} + \frac{1}{\lambda} \|J_{\mu,\alpha\beta}(t, x) - J_{\lambda,\alpha\beta}(t, x)\|_{\eta,\xi} \\
&= \frac{\mu}{\lambda} \|P_{\mu,\alpha\beta}(t, x)\|_{\eta,\xi} + \frac{1}{\lambda} \left\| J_{\mu,\alpha\beta}(t, x) - J_{\mu,\alpha\beta}\left(t, \frac{\mu}{\lambda} x + \frac{\lambda - \mu}{\lambda} J_{\lambda,\alpha\beta}(t, x)\right)\right\|_{\eta,\xi}
\end{aligned}$$

[using (i) above]

$$\leq \frac{\mu}{\lambda} \|P_{\mu,\alpha\beta}(t, x)\|_{\eta,\xi} + \frac{1}{\lambda} \left\| x - \frac{\mu}{\lambda} x - \frac{\lambda - \mu}{\lambda} J_{\lambda,\alpha\beta}(t, x)\right\|_{\eta,\xi}$$

$$= \frac{\mu}{\lambda} \|P_{\mu,\alpha\beta}(t, x)\|_{\eta,\xi} + \frac{\lambda - \mu}{\lambda} \left\| \frac{x - J_{\lambda,\alpha\beta}(t, x)}{\lambda} \right\|_{\eta,\xi}$$

$$= \frac{\mu}{\lambda} \|P_{\mu,\alpha\beta}(t, x)\|_{\eta,\xi} + \frac{\lambda - \mu}{\lambda} \|P_{\lambda,\alpha\beta}(t, x)\|_{\eta,\xi}$$

whence

$$|P_\lambda(t, x)(\eta, \xi)| \leq |P_\mu(t, x)(\eta, \xi)|, \qquad 0 < \mu \leq \lambda$$

for all η, $\xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, α, $\beta \in L^\infty_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$, u, $v \in \mathbb{D}$. This concludes the proof. ■

2.6. Spaces of Sesquilinear-Forms-Valued Maps

We shall employ certain spaces of maps whose values are sesquilinear forms on $\mathbb{D} \underline{\otimes} \mathbb{E}$.

Let $I \subseteq \mathbb{R}_+$ be a subinterval and $L^0(I, \mathbb{D} \underline{\otimes} \mathbb{E})$ the set of all sesq($\mathbb{D} \underline{\otimes} \mathbb{E}$)-valued maps on I. Now, $L^0(I, \mathbb{D} \underline{\otimes} \mathbb{E})$ acquires the structure of a linear space if the linear combination $\alpha u + \beta v$ (α, $\beta \in \mathbb{C}$) of u, $v \in L^0(I, \mathbb{D} \underline{\otimes} \mathbb{E})$ is defined by

$$(\alpha u + \beta v)(t)(\eta, \xi) = \alpha u(t)(\eta, \xi) + \beta v(t)(\eta, \xi)$$

$t \in I$, η, $\xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$. Observe also that every $\tilde{\mathcal{A}}$-valued p on I is in $L^0(I, \mathbb{D} \underline{\otimes} \mathbb{E})$, since p may be identified with the map whose value at $t \in I$ is the sesquilinear form $(\eta, \xi) \mapsto \langle \eta, p(t)\xi \rangle$, η, $\xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$.

In addition to $L^0(I, \mathbb{D} \underline{\otimes} \mathbb{E})$, we introduce the spaces $M(I, \mathbb{D} \underline{\otimes} \mathbb{E})$, $L^p(I, \mathbb{D} \underline{\otimes} \mathbb{E})$, $L^p_{\mathrm{loc}}(I, \mathbb{D} \underline{\otimes} \mathbb{E})$, $1 \leq p \leq \infty$, and $C(I, \mathbb{D} \underline{\otimes} \mathbb{E})$, which are defined as follows:

$$M(I, \mathbb{D} \underline{\otimes} \mathbb{E}) = \{z \in L^0(I, \mathbb{D} \underline{\otimes} \mathbb{E}) \text{: the map } t \mapsto z(t)(\eta, \xi),\ t \in I, \text{ is Lebesgue measurable for arbitrary } \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$$

$$L^p(I, \mathbb{D} \underline{\otimes} \mathbb{E}) = \{z \in L^0(I, \mathbb{D} \underline{\otimes} \mathbb{E}) \text{: the map } t \mapsto z(t)(\eta, \xi),\ t \in I, \text{ is in } L^p(I), \text{ for arbitrary } \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$$

$$L^p_{\mathrm{loc}}(I, \mathbb{D} \underline{\otimes} \mathbb{E}) = \{z \in L^0(I, \mathbb{D} \underline{\otimes} \mathbb{E}) \text{: the map } t \mapsto z(t)(\eta, \xi),\ t \in I, \text{ is in } L^p_{\mathrm{loc}}(I), \text{ for arbitrary } \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$$

$$C(I, \mathbb{D} \underline{\otimes} \mathbb{E}) = \{z \in L^0(I, \mathbb{D} \underline{\otimes} \mathbb{E}) \text{: the map } t \mapsto z(t)(\eta, \xi),\ t \in I, \text{ is in } C(I), \text{ for arbitrary } \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$$

These are locally convex spaces: the topologies τ_p of $L^p(I, \mathbb{D} \underline{\otimes} \mathbb{E})$, $\tau_{p,\text{loc}}$ of $L^p_{\text{loc}}(I, \mathbb{D} \underline{\otimes} \mathbb{E})$, and τ_{con} of $C(I, \mathbb{D} \underline{\otimes} \mathbb{E})$ are generated by the seminorms

$$\tau_p: \quad \{\|\cdot\|_{p,\eta\xi}: \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$$

$$\text{with} \quad \|z\|_{p,\eta\xi} = \left(\int_I dt\, |z(t)(\eta, \xi)|^p\right)^{1/p}$$

$$\tau_{p,\text{loc}}: \quad \{\|\cdot\|_{p,K\eta\xi}: K = \text{compact subset of } I,\ \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$$

$$\text{with} \quad \|z\|_{p,K\eta\xi} = \left(\int_K dt\, |z(t)(\eta, \xi)|^p\right)^{1/p}$$

$$\tau_{\text{con}}: \quad \{\|\cdot\|_{\text{con},\eta\xi}: \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$$

$$\text{with} \quad \|z\|_{\text{con},\eta\xi} = \sup_{t \in I} |z(t)(\eta, \xi)|$$

respectively.

Definition. A member $z \in L^0(I, \mathbb{D} \underline{\otimes} \mathbb{E})$ is:

(i) *Absolutely continuous* if the map $t \mapsto z(t)(\eta, \xi)$ is absolutely continuous for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$.

(ii) Of *bounded variation* if

$$\sup\left(\sum_{j=1}^{n} |z(t_j)(\eta, \xi) - z(t_{j-1})(\eta, \xi)|\right) < \infty$$

for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, where the supremum is taken over all partitions $\{t_j\}_{j=0}^{n}$ of I.

(iii) Of *essentially bounded variation* if z is equal almost everywhere to some member of $L^0(I, \mathbb{D} \underline{\otimes} \mathbb{E})$ of bounded variation.

Remark. The following result will be employed below.

Proposition 2.3. Let $\overline{T} > 0$, $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, $h: [0, \overline{T}] \to \tilde{\mathcal{A}}$ a member of $L^1([0, \overline{T}], \mathbb{D} \underline{\otimes} \mathbb{E})$, and $\epsilon > 0$. Then, there is a partition $\Delta_N = \{0 = t_0 < t_1 < \cdots < t_N \leq \overline{T}\}$ such that:

(i) $\max_{1 \leq k \leq N} \leq \epsilon$ and $\overline{T} - t_N \leq \epsilon$.

(ii) The step function $g: (0, t_N) \to \tilde{\mathcal{A}}$ defined by

$$g(t) = h(t_k) \qquad \text{for} \quad t \in (t_{k-1}, t_k], \quad k = 1, 2, \ldots, N$$

satisfies

$$\|g - h\|^{(N)}_{1,\eta\xi} \leq \epsilon$$

where $\{\|\cdot\|^{(N)}_{1,\eta\xi}: \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$ is the family of seminorms of $L^1([0, t_n], \mathbb{D} \underline{\otimes} \mathbb{E})$.

Proof. Either (a) $\|h\|_{1,\eta\xi}^{(N)} = 0$ or (b) $\|h\|_{1,\eta\xi}^{(N)} \neq 0$. If (a) holds, then there is nothing to prove. In case (b) holds, one argues as in Lemma 4.1 of Evans (1977) to conclude the proof. ■

3. INCLUSIONS OF EVOLUTION TYPE

Let I be a subinterval of $\mathbb{R}_+$. As in Ekhaguere (1995), a map $X: I \to \tilde{\mathcal{A}}$ will be called a *stochastic process* indexed by I. If $X(t) \in \tilde{\mathcal{A}}_t$ for each $t \in I$, then X is *adapted*. The set of all adapted stochastic processes indexed by I will be denoted by $\mathrm{Ad}(I, \tilde{\mathcal{A}})$, with $\mathrm{Ad}(\mathbb{R}_+, \tilde{\mathcal{A}})$ written simply as $\mathrm{Ad}(\tilde{\mathcal{A}})$. As clarified in Section 2.6, $\mathrm{Ad}(I, \tilde{\mathcal{A}})$ is a subset of $L^0(I, \mathbb{D} \underline{\otimes} \mathbb{E})$ in a natural way. Therefore, the notions introduced in that subsection apply also to the members of $\mathrm{Ad}(I, \tilde{\mathcal{A}})$.

3.1. Quantum Stochastic Integration

We fix $f, g \in L^\infty_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$ and $\pi \in L^\infty_{B(\gamma),\mathrm{loc}}(\mathbb{R}_+)$ throughout the paper. To these maps correspond the adapted stochastic processes A_f, A_g^+, and Λ_π defined by

$$A_f(t) = a(f\chi_{[0,t)}) \otimes 1'$$

$$A_g^+(t) = a^+(g\chi_{[0,t)}) \otimes 1'$$

$$\Lambda_\pi(t) = \lambda(\pi\chi_{[0,t)}) \otimes 1'$$

where $t \in \mathbb{R}_+$, $a(f)$, $a^+(g)$, and $\lambda(\pi)$ are the *annihilation, creation,* and *gauge* operators of quantum field theory (Hudson and Parthasarathy, 1984) and χ_C is the indicator function of the Borel set $C \subseteq \mathbb{R}_+$.

If $p, q, u, v \in L^2_{\mathrm{loc}}(I, \mathbb{D} \underline{\otimes} \mathbb{E})$, $I \subseteq \mathbb{R}_+$, then in the sequel, we interpret the stochastic integral

$$\int_I (p(s)\, d\Lambda_\pi(s) + q(s)\, dA_f(s) + u(s)\, dA_g^+(s) + v(s)\, ds)$$

in the sense of Hudson and Parthasarathy (1984).

3.2. Stochastic Differential Inclusions

For $\mathcal{M}$ a closed subset of $\tilde{\mathcal{A}}$ and $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, we define $\|\mathcal{M}\|_{\eta,\xi}$ in the sequel as in Ekhaguere (1992), p. 2006.

Let $I \subseteq \mathbb{R}_+$ be a subinterval. In the subsequent discussion, we deal mainly with *multivalued stochastic processes* indexed by I. These are maps $\Phi: I \to 2^{\tilde{\mathcal{A}}}$ with closed values. If $\Phi(t) \subseteq \tilde{\mathcal{A}}_t$ for each $t \in I$, then Φ is called *adapted*. Suppose $\Phi: I \to 2^{\tilde{\mathcal{A}}}$ is adapted. If $t \mapsto \|\Phi(t)\|_{\eta,\xi}$, $t \in I$, is in

$L^2_{\text{loc}}(I)$ for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, then Φ is *locally absolutely square integrable*. The set of all such multivalued stochastic processes will be denoted by $L^2_{\text{loc}}(I, \tilde{\mathcal{A}})_{\text{mvs}}$, with $L^2_{\text{loc}}(\mathbb{R}_+, \tilde{\mathcal{A}})_{\text{mvs}}$ written simply as $L^2_{\text{loc}}(\tilde{\mathcal{A}})_{\text{mvs}}$, and the notation $L^2_{\text{loc}}(I \times \tilde{\mathcal{A}})_{\text{mvs}}$ will be reserved for the set of multifunctions Φ: $I \times \tilde{\mathcal{A}} \to 2^{\tilde{\mathcal{A}}}$, with closed values, such that the map $t \mapsto \Phi(t, X(t))$, $t \in I$, is in $L^2_{\text{loc}}(I, \tilde{\mathcal{A}})_{\text{mvs}}$ for arbitrary $X \in \text{Ad}(I, \tilde{\mathcal{A}}) \cap L^2_{\text{loc}}(I, \mathbb{D} \underline{\otimes} \mathbb{E})$.

In the sequel, $T > 0$, E, F, G, H are in $L^2_{\text{loc}}([0, T] \times \tilde{\mathcal{A}})_{\text{mvs}}$, p: $[0, T] \to \tilde{\mathcal{A}}$ is an adapted stochastic process, and the following *initial value stochastic differential inclusion* is introduced as in Ekhaguere (1995), Section 5.1:

$$\begin{aligned} dX(t) \in {} & -(E(t, X(t))\, d\Lambda_\pi(t) + F(t, X(t))\, dA_f(t) + G(t, X(t))\, dA_g^+(t) \\ & + H(t, X(t))\, dt) + p(t)\, dt, \qquad \text{almost all} \quad t \in (0, T] \quad (3.2.1)_X \\ X(0) = {} & x_0 \qquad \text{for some} \quad x_0 \in \tilde{\mathcal{A}} \end{aligned}$$

As in Ekhaguere (1995), we recast this inclusion as follows. For $\alpha, \beta \in L^\infty_{\Upsilon,\text{loc}}(\mathbb{R}_+)$, define the multifunction $P_{\alpha\beta}$: $\mathbb{R}_+ \times \tilde{\mathcal{A}} \to 2^{\tilde{\mathcal{A}}}$ by

$$P_{\alpha\beta}(t, x) = \mu_{\alpha\beta}(t)E(t, x) + \nu_\beta(t)F(t, x) + \sigma_\alpha(t)G(t, x) + H(t, x)$$

where $\mu_{\alpha\beta}(t) = \langle \alpha(t), \pi(t)\beta(t)\rangle_\Upsilon$, $\nu_\beta(t) = \langle f(t), \beta(t)\rangle_\Upsilon$, and $\sigma_\alpha(t) = \langle \alpha(t), g(t)\rangle_\Upsilon$, $(t, x) \in \mathbb{R}_+ \times \tilde{\mathcal{A}}$, and $\langle \cdot, \cdot \rangle_\Upsilon$ is the inner product of the Hilbert space Υ. This gives rise to the multifunction

$$P: \quad [0, T] \times \tilde{\mathcal{A}} \to 2^{\text{sesq}(\mathbb{D}\underline{\otimes}\mathbb{E})}$$

defined by

$$\begin{aligned} P(t, x)(\eta, \xi) &= \langle \eta, P_{\alpha\beta}(t, x)\xi \rangle \\ &= \{\langle \eta, p_{\alpha\beta}(t, x)\xi \rangle : p_{\alpha\beta}(t, x) \in P_{\alpha\beta}(t, x)\} \end{aligned} \tag{3.2.2}$$

$(t, x) \in \mathbb{R}_+ \times \tilde{\mathcal{A}}$, $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^\infty_{\Upsilon,\text{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$. Then, by Theorem 6.2 of Ekhaguere (1992), the initial value stochastic differential inclusion $(3.2.1)_X$ is equivalent to the following initial value nonclassical differential inclusion:

$$\begin{aligned} & \frac{d}{dt}\langle \eta, X(t)\xi \rangle \in -P(t, X(t))(\eta, \xi) + \langle \eta, p(t)\xi \rangle \qquad \text{almost all} \quad t \in (0, T] \\ & X(0) = x_0 \in \tilde{\mathcal{A}} \qquad\qquad (3.2.1)_P \end{aligned}$$

for arbitrary $(\eta, \xi) \in (\mathbb{D} \underline{\otimes} \mathbb{E})^2$.

Definition. A map φ: $[0, T] \to \tilde{\mathcal{A}}$ is a *solution* of Problem $(3.2.1)_X$ if it is adapted, absolutely continuous, and satisfies

$$d\varphi(t) \in -(E(t, \varphi(t))\, d\Lambda_\pi(t) + F(t, \varphi(t))\, dA_f(t) + G(t, \varphi(t))\, dA_g^+(t)$$
$$+ H(t, \varphi(t))\, dt) + p(t)\, dt \qquad \text{almost all} \quad t \in (0, T]$$
$$\varphi(0) = x_0 \in \tilde{\mathscr{A}}$$

Definition. Problem $(3.2.1)_X$ will be said to be of *hypermaximal monotone type* or *evolution type* if the multifunction P in $(3.2.1)_P$ is such that $\mathcal{P} = P \otimes 1$ lies in $\text{Hypmax}(\mathbb{R}_+ \times \tilde{\mathscr{A}})$. Similarly, Problem $(3.2.1)_X$ is *Lipschitzian* if P is Lipschitzian, as explained in Ekhaguere (1992).

Remark. 1. In Ekhaguere (1992), we established the existence of a solution of a *Lipschitzian* stochastic differential inclusion and proved a *Relaxation Theorem* giving the relationship between the solutions of such an inclusion and those of its convexification. In Ekhaguere (1995) we proved that a stochastic differential inclusion of hypermaximal monotone type possesses a *unique* adapted solution, obtainable as the limit of the unique adapted solutions of a one-parameter family of *Lipschitzian* stochastic differential *equations*.

2. In this paper, we consider quantum stochastic differential inclusions involving multifunctions $\{P(t, \cdot): t \in [0, T], T > 0\}$ that are hypermaximal monotone for almost all $t \in [0, T]$ and study the problem of existence of *evolution operators* corresponding to them.

3.3. The Hypotheses on P and p

Let $T > 0$. In the sequel, we shall impose some of the following hypotheses on the multifunction P and the adapted stochastic process p.

(S_0) For almost all $t \in [0, T]$, the multifunction $\mathcal{P}(t, \cdot) = P(t, \cdot) \otimes 1$ is hypermaximal monotone, i.e., $\mathcal{P}(t, \cdot) = P(t, \cdot) \otimes 1 \in \text{Hypmax}([0, T] \times \tilde{\mathscr{A}})$.

(S_1) (a) There exist $\lambda_0 > 0$, a member $h: [0, T] \to \tilde{\mathscr{A}}$ of $L^1([0, T], \mathbb{D} \underline{\otimes} \mathbb{E})$, and a nondecreasing continuous function $L: \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|P_\lambda(t, x)(\eta, \xi) - P_\lambda(s, x)(\eta, \xi)| \leq \|h(t) - h(s)\|_{\eta,\xi} L(\|x\|_{\eta,\xi})$$

for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, $0 < \lambda \leq \lambda_0$, $x \in \tilde{\mathscr{A}}$, and almost all $0 \leq s, t \leq T$.

(b) The map $p: [0, T] \to \tilde{\mathscr{A}}$ is in $L^1([0, T], \mathbb{D} \underline{\otimes} \mathbb{E})$.

(S_2) (a) There exist $\lambda_0 > 0$, a member $h: [0, T] \to \tilde{\mathscr{A}}$ of $M(I, \mathbb{D} \underline{\otimes} \mathbb{E})$ of essentially bounded variation, and a nondecreasing continuous function $L: \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|P_\lambda(t, x)(\eta, \xi) - P_\lambda(s, x)(\eta, \xi)| \leq \|h(t) - h(s)\|_{\eta,\xi} L(\|x\|_{\eta,\xi})$$
$$(1 + |P_\lambda(t, x)(\eta, \xi)|)$$

for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, $0 < \lambda \leq \lambda_0$, $x \in \tilde{\mathscr{A}}$, and almost all $0 \leq s, t \leq T$.

(b) The map p: $[0, T] \rightarrow \tilde{\mathcal{A}}$ is in $M([0, T], \mathbb{D} \underline{\otimes} \mathbb{E})$ and of essentially bounded variation.

Remark. 1. Compare the inequalities in (S_1)(a) and (S_2)(a) with the inequality in Proposition 5.4 of Ekhaguere (1995). Using the latter inequality and with $p \equiv 0$, we proved in that paper that $(3.2.1)_X$ possesses a unique adapted solution.

2. Throughout the paper, a reference to (S_1) [resp. (S_2)] always means the combination (S_1)(a) + (S_1)(b) [resp. (S_2)(a) + (S_2)(b)].

Notation. Let $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ and $(t, x) \in [0, T] \times D(P(t, \cdot))$. By Proposition 2.2(ii), the map $\lambda \mapsto P_\lambda(t, x)(\eta, \xi)$ is monotone decreasing for all $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ and $(t, x) \in [0, T] \times \tilde{\mathcal{A}}$.

In the sequel, $|P(t, x)|_{\eta\xi}$ and $\hat{D}(P(t, \cdot))$ are defined by

$$|P(t, x)|_{\eta\xi} = \lim_{\lambda \searrow 0} |P_\lambda(t, x)(\eta, \xi)|, \qquad \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}, \quad (t, x) \in [0, T] \times \tilde{\mathcal{A}}$$

and

$$\hat{D}(P(t, \cdot)) = \{x \in \tilde{\mathcal{A}}: |P(t, x)|_{\eta\xi} < \infty \text{ for } \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}, \qquad t \in [0, T]$$

We write $\overline{D(P(t, \cdot))}$ for the closure of $D(P(t, \cdot))$.

Notice that $D(P(t, \cdot)) \subseteq \hat{D}(P(t, \cdot)) \subseteq \overline{D(P(t, \cdot))}$. The set $\hat{D}(P(t, \cdot))$ is called the *generalized domain* of $P(t, \cdot)$.

Remark. We shall repeatedly employ the following immediate consequence of the above hypotheses.

Proposition 3.1. Suppose (S_0) holds for P. If, additionally, either (S_1) or (S_2) also holds, then the multifunctions $t \mapsto \hat{D}(P(t, \cdot))$ and $t \mapsto \overline{D(P(t, \cdot))}$ from $[0, T]$ to $2^{\tilde{\mathcal{A}}}$ are constant almost everywhere.

Proof. Let (S_0) and (S_1) hold. Let $[0, T]_h$ be the subset of $[0, T]$ on which h is defined and $s, t \in [0, T]_h$. Then

$$|P_\lambda(t, x)(\eta, \xi)| - |P_\lambda(s, x)(\eta, \xi)| \leq \|h(t) - h(s)\|_{\eta,\xi} L(\|x\|_{\eta,\xi}) \tag{3.3.1}$$

whence

$$|P_\lambda(s, x)(\eta, \xi)| \leq |P_\lambda(t, x)(\eta, \xi)| + \|h(t) - h(s)\|_{\eta,\xi} L(\|x\|_{\eta,\xi})$$

for $x \in D(P(t, \cdot))$ and $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$. It follows that $x \in \hat{D}(P(s, \cdot))$ whenever $x \in \hat{D}(P(t, \cdot))$, i.e., $\hat{D}(P(t, \cdot)) \subseteq \hat{D}(P(s, \cdot))$. Similarly, one gets that $\hat{D}(P(s, \cdot)) \subseteq \hat{D}(P(t, \cdot))$. Hence, $\hat{D}(P(s, \cdot)) = \hat{D}(P(t, \cdot))$, for arbitrary $s, t \in [0, T]_h$, showing that the multifunction $t \mapsto \hat{D}(P(t, \cdot))$ is constant almost everywhere. This also proves that the multifunctions $t \mapsto D(P(t, \cdot))$ and $t \mapsto$

$\overline{D(P(t,\ \cdot))}$ are constant almost everywhere. The proof is similar when we assume that (S_0) and (S_2) hold.

Remark. Let the hypotheses of Proposition 3.1 hold. If $t \mapsto P(t,\ \cdot)$ and h are defined at a point $t_0 \in [0, T]$ and x_δ is a fixed member of $\hat{D}(P(t_0,\ \cdot))$, then for $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, (3.3.1) gives

$$|P_\lambda(s, x_\delta)(\eta, \xi)| \leq k^1_{\eta\xi} + k^2_{\eta\xi}\|h\|_{\eta,\xi} \qquad \text{almost all} \quad s \in [0, T] \tag{3.3.2}$$

where $k^1_{\eta\xi}$ and $k^2_{\eta\xi}$ depend on $|P(t_0, x_\delta)|_{\eta\xi}$, $\|x_\delta\|_{\eta,\xi}$, and $\|h(t_0)\|_{\eta,\xi}$.

4. ITERATIVE ESTIMATES

In this section, we introduce discrete approximations of Problem $(3.2.1)_P$ and compare the solutions of two such approximation schemes.

In the sequel, partitions are assumed to consist of points not falling into the null set where (S_0), (S_1), or (S_2) fail.

4.1. The Approximation Schemes

Let $\{s_j\}_{j=1}^M$, $\{t_k\}_{k=1}^N$ be two partitions of $[0, T]$ and $\{p_j\}_{j=1}^M$, $\{q_k\}_{k=1}^N$ two subsets of members of $\tilde{\mathcal{A}}$. Then, we consider the solutions $\{x_j\}_{j=1}^M$ and $\{y_k\}_{k=1}^N$, consisting of members of $\tilde{\mathcal{A}}$, of the discrete schemes

$$\frac{\langle \eta, (x_j - x_{j-1})\xi \rangle}{s_j - s_{j-1}} \in -P(s_j, x_j)(\eta, \xi) + \langle \eta, p_j\xi \rangle\rangle, \qquad j = 1, 2, \ldots, M \tag{4.1.1$_j$}$$

and

$$\frac{\langle \eta, (y_k - y_{k-1})\xi \rangle}{t_k - t_{k-1}} \in -P(t_k, y_k)(\eta, \xi) + \langle \eta, q_k\xi \rangle, \qquad k = 1, 2, \ldots, N \tag{4.1.1$_k$}$$

for $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$.

Introduce the notation

$$\gamma_j = s_j - s_{j-1}, \qquad \delta_k = t_k - t_{k-1}$$

$$\sigma_{jk} = \frac{\gamma_j\delta_k}{\gamma_j + \delta_k}, \qquad j = 1, 2, \ldots, M; \quad k = 1, 2, \ldots, N$$

$$l^1_{\eta\xi} = \max_{0\leq j\leq M} \{\|x_j\|_{\eta,\xi}\}$$

$$l^2_{\eta\xi} = \max_{0\leq j\leq M}\left\{\left\|p_j + \frac{x_{j-1} - x_j}{\gamma_j}\right\|_{\eta,\xi}\right\}$$

for $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$.

Remark. (i) Using the notation of Section 2.5, we can rewrite the inclusions $(4.1.1)_j$ and $(4.1.1)_k$ equivalently as follows:

$$J_{\lambda\sigma_{jk},\alpha\beta}\left(s_j, x_j + \lambda\sigma_{jk}\left(p_j + \frac{x_{j-1} - x_j}{\gamma_j}\right)\right) = x_j \qquad (4.1.2)_j$$

$$J_{\lambda\sigma_{jk},\alpha\beta}\left(t_k, y_k + \lambda\sigma_{jk}\left(q_k + \frac{y_{k-1} - y_k}{\delta_k}\right)\right) = y_k \qquad (4.1.2)_k$$

for all $\lambda > 0$, $\alpha, \beta \in L^{\infty}_{\Upsilon,\mathrm{loc}}(\mathbb{R}_+)$, $j = 1, 2, \ldots, M$; $k = 1, 2, \ldots, N$. One sees this in the case of $(4.1.2)_j$ thus. Let $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^{\infty}_{\Upsilon,\mathrm{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$. Then, from $(4.1.1)_j$, one gets

$$\frac{x_j - x_{j-1}}{\gamma_j} \in -P_{\alpha\beta}(s_j, x_j) + p_j$$

whence

$$x_j \in x_j + \lambda\sigma_{jk} P_{\alpha\beta}(s_j, x_j) - \lambda\sigma_{jk}\left(p_j + \frac{x_{j-1} - x_j}{\gamma_j}\right)$$

Equation $(4.1.2)_j$ follows from this. The proof of $(4.1.2)_k$ is analogous.

(ii) Using $(4.1.2)_j$, one gets

$$\begin{aligned}
&\lambda\sigma_{jk} P_{\lambda\sigma_{jk},\alpha\beta}\left(s_j, x_j + \lambda\sigma_{jk}\left(p_j + \frac{x_{j-1} - x_j}{\gamma_j}\right)\right)\\
&= x_j + \lambda\sigma_{jk}\left(p_j + \frac{x_{j-1} - x_j}{\gamma_j}\right) - J_{\lambda\sigma_{jk},\alpha\beta}\left(s_j, x_j + \lambda\sigma_{jk}\left(p_j + \frac{x_{j-1} - x_j}{\gamma_j}\right)\right)\\
&= \lambda\sigma_{jk}\left(p_j + \frac{x_{j-1} - x_j}{\gamma_j}\right)
\end{aligned}$$

showing that

$$\begin{aligned}
&\left|P_{\lambda\sigma_{jk}}\left(s_j, x_j + \lambda\sigma_{jk}\left(p_j + \frac{x_{j-1} - x_j}{\gamma_j}\right)\right)(\eta, \xi)\right|\\
&= \left\|P_{\lambda\sigma_{jk},\alpha\beta}\left(s_j, x_j + \lambda\sigma_{jk}\left(p_j + \frac{x_{j-1} - x_j}{\gamma_j}\right)\right)\right\|_{\eta,\xi}\\
&= \left\|p_j + \frac{x_{j-1} - x_j}{\gamma_j}\right\|_{\eta,\xi}\\
&\le l^2_{\eta\xi} \qquad (4.1.3)
\end{aligned}$$

for $\lambda > 0$, $j = 1, 2, \ldots, M$; $k = 1, 2, \ldots, N$, and arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^{\infty}_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$.

(iii) We now compare the solutions of the discrete schemes $(4.1.1)_j$ and $(4.1.1)_k$.

Proposition 4.1. Assume that

(i) P satisfies (S_0) and either (S_1) or (S_2).

(ii) $(4.1.1)_j$ and $(4.1.1)_k$ hold for $j \le j \le M$, $1 \le k \le N$.

Then, for $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^{\infty}_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$, there is a nonnegative constant $k_{\eta\xi}$, depending on $l^1_{\eta\xi}$ if (S_1) holds, but otherwise on both $l^1_{\eta\xi}$ and $l^2_{\eta\xi}$ if (S_2) holds, such that

$$\|x_j - y_k\|_{\eta,\xi} \le \frac{\delta_k}{\gamma_j + \delta_k}\|x_{j-1} - y_k\|_{\eta,\xi} + \frac{\gamma_j}{\gamma_j + \delta_k}\|x_j - y_{k-1}\|_{\eta,\xi}$$

$$+ \frac{\gamma_j\delta_k}{\gamma_j + \delta_k}(k_{\eta\xi}\|h(s_j) - h(t_k)\|_{\eta,\xi} + [p_j - q_k, x_j - y_k]_{\eta\xi})$$

$1 \le j \le M$, $1 \le k \le N$.

Proof. We shall prove the result under the hypotheses (S_0) and (S_2): the proof is simpler when (S_0) and (S_1) are assumed.

Let $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^{\infty}_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$. Then,

$$\|x_j - y_k\|_{\eta,\xi}$$

$$= \left\| J_{\lambda\sigma_{jk},\alpha\beta}\left(s_j, x_j + \lambda\sigma_{jk}\left(p_j + \frac{x_{j-1} - x_j}{\gamma_j}\right)\right)\right\|_{\eta,\xi}$$

$$- \left\| J_{\lambda\sigma_{jk},\alpha\beta}\left(t_k, y_k + \lambda\sigma_{jk}\left(q_k + \frac{y_{k-1} - y_k}{\delta_k}\right)\right)\right\|_{\eta,\xi}$$

[by $(4.2.1)_j$ and $(4.2.1)_k$]

$$\le \left\| J_{\lambda\sigma_{jk},\alpha\beta}\left(t_k, x_j + \lambda\sigma_{jk}\left(p_j + \frac{x_{j-1} - x_j}{\gamma_g}\right)\right)\right\|_{\eta,\xi}$$

$$- \left\| J_{\lambda\sigma_{jk},\alpha\beta}\left(t_k, y_k + \lambda\sigma_{jk}\left(q_k + \frac{y_{k-1} - y_k}{\delta_k}\right)\right)\right\|_{\eta,\xi}$$

$$+ \left\| J_{\lambda\sigma_{jk},\alpha\beta}\left(s_j, x_j + \lambda\sigma_{jk}\left(p_j + \frac{x_{j-1} - x_j}{\gamma_j}\right)\right)\right.$$

$$- J_{\lambda\sigma_{jk},\alpha\beta}\left(t_k, x_j + \lambda\sigma_{jk}\left(p_j + \frac{x_{j-1} - x_j}{\gamma_j}\right)\right)\Bigg\|_{\eta,\xi}$$

$$\leq \left\|x_j + \lambda\sigma_{jk}\left(p_j + \frac{x_{j-1} - x_j}{\gamma_j}\right) - \left(y_k + \lambda\sigma_{jk}\left(q_k + \frac{y_{k-1} - y_k}{\delta_k}\right)\right)\right\|_{\eta,\xi}$$

$$+ \lambda\sigma_{jk}\Bigg\|P_{\lambda\sigma_{jk},\alpha\beta}\left(s_j, x_j + \lambda\sigma_{jk}\left(p_j + \frac{x_{j-1} - x_j}{\gamma_j}\right)\right)$$

$$- P_{\lambda\sigma_{jk},\alpha\beta}\left(t_k, x_j + \lambda\sigma_{jk}\left(p_j + \frac{x_{j-1} - x_j}{\gamma_j}\right)\right)\Bigg\|_{\eta,\xi}$$

$$\leq \left\|x_j + \lambda\sigma_{jk}\left(p_j + \frac{x_{j-1} - x_j}{\gamma_j}\right) - \left(y_k + \lambda\sigma_{jk}\left(q_k + \frac{y_{k-1} - y_k}{\delta_k}\right)\right)\right\|_{\eta,\xi}$$

$$+ \lambda\sigma_{jk}\|h(s_j) - h(t_k)\|_{\eta,\xi} L\left(\left\|x_j + \lambda\sigma_{jk}\left(p_j + \frac{x_{j-1} - x_j}{\gamma_j}\right)\right\|_{\eta,\xi}\right)(1 + l^2_{\eta\xi})$$

[by (S_2) and (4.1.3)]

Hence

$$\|x_j - y_k\|_{\eta,\xi}$$
$$= \lambda\|x_j - y_k\|_{\eta,\xi} + (1 - \lambda)\|x_j - y_k\|_{\eta,\xi}$$
$$\leq \|(1 - \lambda)(x_j - y_k) + \lambda\sigma_{jk}(p_j - q_k)\|_{\eta,\xi} + \frac{\lambda\gamma_j}{\gamma_j + \delta_k}\|x_j - y_{k-1}\|_{\eta,\xi}$$
$$+ \lambda\sigma_{jk}\|h(s_j) - h(t_k)\|_{\eta,\xi} L\left(\left\|x_j + \lambda\sigma_{jk}\left(p_j + \frac{x_{j-1} - x_j}{\gamma_j}\right)\right\|_{\eta,\xi}\right)(1 + l^2_{\eta\xi})$$

whence, setting $\lambda/(1 - \lambda) = \epsilon$,

$$\|x_j - y_k\|_{\eta,\xi}$$
$$\leq \frac{\delta_k}{\gamma_j + \delta_k}\|x_{j-1} - y_k\|_{\eta,\xi} + \frac{\gamma_j}{\gamma_j + \delta_k}\|x_j - y_{k-1}\|_{\eta,\xi}$$
$$+ \frac{\|x_j - y_k + \epsilon\sigma_{jk}(p_j - q_k)\|_{\eta,\xi} - \|x_j - y_k\|_{\eta,\xi}}{\epsilon}$$
$$+ \sigma_{jk}\|h(s_j) - h(t_k)\|_{\eta,\xi} L\left(\left\|x_j + \lambda\sigma_{jk}\left(p_j + \frac{x_{j-1} - x_j}{\gamma_j}\right)\right\|_{\eta,\xi}\right)(1 + l^2_{\eta\xi})$$

The assertion follows from this by letting $\epsilon \searrow 0$ and using the continuity of L. ∎

5. CONVERGENCE THEOREMS

In this section, we prove the main results about the convergence of the approximation schemes introduced in the previous section.

Theorem 5.1. Let $\overline{T} > 0$, $x_0 \in \overline{D(P)}$, and $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^{\infty}_{\mathbb{Y},\mathrm{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$. Assume (S_0) and (S_1). Then, for every $T \in (0, \overline{T})$, there exist a sequence $\{\Delta_n\}_{n\in\mathbb{N}}$ of partitions $\Delta_n \equiv \{0 = t_0^n < t_1^n < \cdots < t_{N(n)}^n \equiv T(n)\}$ and sequences $\{x_k^n\}_{k=0}^{N(n)}$, $\{p_k^n\}_{k=1}^{N(n)}$ of members of $\tilde{\mathcal{A}}$ such that:

(i) $t \mapsto P(t, \cdot)$ and $t \mapsto p(t)$ are defined for each positive partition point t_k^n.

(ii) We have

$$\frac{\langle \eta, (x_k^n - x_{k-1}^n)\xi\rangle}{t_k^n - t_{k-1}^n} \in -P(t_k^n, x_k^n)(\eta, \xi) + \langle \eta, p_k^n \xi\rangle$$

$k = 1, 2, \ldots, N(n)$; $n = 1, 2, \ldots$.

(iii) $\lim_{n\to\infty} \max_{1\le k\le N(n)}(t_k^n - t_{k-1}^n) = 0$.

(iv) $T \le T(n) \le \overline{T}$, $n = 1, 2, \ldots$.

(v) $x_0^n = x_0$, $n = 1, 2, \ldots$.

(vi) The step functions p^n defined by

$$p^n(t) = p_k^n \qquad \text{for} \quad t \in (t_{k-1}^n, t_k^n]$$

converge to p in $L^1([0, T], \mathbb{D} \underline{\otimes} \mathbb{E})$.

(vii) The step functions φ^n defined by

$$\varphi^n(t) = x_k^n \qquad \text{for} \quad t \in (t_{k-1}^n, t_k^n]$$

converge in $C([0, T], \mathbb{D} \underline{\otimes} \mathbb{E})$ to a stochastic process φ_∞: $[0, T] \to \tilde{\mathcal{A}}$.

Proof. Let $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^{\infty}_{\mathbb{Y},\mathrm{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$. Let $[0, \overline{T}]_{\mathrm{def}}$ be the subset of $[0, \overline{T}]$ on which p is defined, let $t \mapsto P(t, \cdot)$ be hypermaximal monotone, and let the inequality of hypothesis (S_1)(a) hold. Then, by Proposition 2.3, there is a sequence $\Delta_n = \{0 = t_0^n < t_1^n < \cdots < t_{N(n)}^n \equiv T(n)\}$ whose positive points all lie in $[0, \overline{T}]_{\mathrm{def}}$, with $T \le T(n) \le \overline{T}$ and

$$\lim_{n\to\infty} \max_{1\le k\le N(n)} (t_k^n - t_{k-1}^n) = 0$$

such that h^n and p^n defined as in the theorem satisfy

$$\lim_{n\to\infty} \|h - h^n\|_{1,\eta\xi}^{(n)} = 0 = \lim_{n\to\infty} \|p - p^n\|_{1,\eta\xi}^{(n)} \tag{5.1.1}$$

$\{\|\cdot\|^{(n)}_{1,\eta\xi}: \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$ is the family of seminorms of $L^1([0, T(n)], \mathbb{D} \underline{\otimes} \mathbb{E})$.

Denote $p(t_k^n)$ by p_k^n and set $x_0^n \equiv x_0$.

With the set $\{t_k^n\}$ and the point x_0^n available, one can form the inclusion

$$\frac{\langle \eta, (x_k^n - x_{k-1}^n)\xi \rangle}{t_k^n - t_{k-1}^n} \in -P(t_k^n, x_k^n)(\eta, \xi) + \langle \eta, p_k^n \xi \rangle$$

$k = 1, 2, \ldots, N(n)$; $n = 1, 2, \ldots$; and solve this approximation scheme by iteration. This proves the claims (i)–(vi).

To prove (vii), let $\delta > 0$ and select $t_0 \in [0, T]$ such that $P(t_0, \cdot)$ and $h(t_0)$ are defined. Denote $\hat{D}(P(t_0, \cdot))$ by $\hat{D}$. Pick $x_\delta \in \hat{D}$ such that $\|x_\delta - x_0\|_{\eta,\xi} \leq \delta$. Then

$$\begin{aligned}
&\|x_k^n - x_\delta\|_{\eta,\xi} \\
&\quad \leq \|J_{\delta_k^n,\alpha\beta}(t_k^n, x_{k-1}^n + \delta_k^n p_k^n) - J_{\delta_k^n,\alpha\beta}(t_k^n, x_\delta)\|_{\eta,\xi} \\
&\qquad + \|J_{\delta_k^n,\alpha\beta}(t_k^n, x_\delta) - x_\delta\|_{\eta,\xi} \\
&\quad \leq \|x_{k-1}^n + \delta_k^n p_k^n - x_\delta\|_{\eta,\xi} + \|\delta_k^n P_{\delta_k^n,\alpha\beta}(t_k^n, x_\delta)\|_{\eta,\xi} \\
&\qquad \text{[by part 1(i) of Theorem 4.1 of Ekhaguere (1995)]} \\
&\quad \leq \|x_{k-1}^n - x_\delta\|_{\eta,\xi} + \delta_k^n \|p_k^n\|_{\eta,\xi} + \delta_k^n |P(t_k^n, x_\delta)|_{\eta\xi} \\
&\qquad \text{[by Proposition 2.2(ii)]}
\end{aligned}$$

This iterative inequality leads to

$$\|x_k^n - x_\delta\|_{\eta,\xi} \leq k^1_{\eta\xi} t_k^n + \sum_{j=1}^{k} \delta_j^n (\|p_j^n\|_{\eta,\xi} + k^2_{\eta\xi} \|h(t_j^n)\|_{\eta,\xi})$$

where use has been made of (3.3.2).

Since

$$\|x_k^n - x_0^n\|_{\eta,\xi} \leq \|x_k^n - x_\delta\|_{\eta,\xi} + \|x_\delta - x_0^n\|_{\eta,\xi}$$

it follows that

$$\begin{aligned}
&\|x_k^n - x_0^n\|_{\eta,\xi} \\
&\quad \leq \|x_\delta - x_0^n\|_{\eta,\xi} + k^1_{\eta\xi} t_k^n + \sum_{j=1}^{k} \delta_j^n (\|p_j^n\|_{\eta,\xi} + k^2_{\eta\xi} \|h(t_j^n)\|_{\eta,\xi}) \\
&\quad \leq \Theta_{\delta,\eta\xi}(t_k^n) + \Pi^n_{\delta,\eta\xi} \qquad (5.1.2)
\end{aligned}$$

with

$$\Theta_{\delta,\eta\xi}(t) = k^1_{\eta\xi}t + \int_0^t d\tau\,(\|p(\tau)\|_{\eta,\xi} + k^2_{\eta\xi}\|h(\tau)\|_{\eta,\xi}) + \|x_\delta - x_0\|_{\eta,\xi}$$

and

$$\Pi^n_{\delta,\eta\xi} \int_0^{T(n)} d\tau\,(\|p^n(\tau) - p(\tau)\|_{\eta,\xi} + k^2_{\eta\xi}\|h^n(\tau) - h(\tau)\|_{\eta,\xi})$$

For fixed δ, $\lim_{n\to\infty} \Pi^n_{\delta,\eta\xi} = 0$, by (5.1.1).

Next, define the grids $\{\Delta_{m,n}: m, n \in \mathbb{N}\}$ by

$$\{\Delta_{m,n} = \{(t^m_j, t^n_k): j = 1, 2, \ldots, N(m); k = 1, 2, \ldots, N(n)\}$$

and the sets

$$\Omega_{m,n} = (0, T(m)] \times (0, T(n)]$$
$$\Omega = (0, T] \times (0, T]$$

Denote the even extension of the continuous function $\Theta_{\delta,\eta\xi}$ to the whole of $[-\overline{T}, \overline{T}]$ again by $\Theta_{\delta,\eta\xi}$, and let $a^{m,n}_{jk,\eta\xi} = \|x^m_j - x^n_k\|_{\eta,\xi}$. By Proposition 4.1 there is a positive number $k_{\eta\xi}$, depending on $\max_{1\le k\le n}(\|x^n_k\|_{\eta,\xi})$, such that

$$a^{m,n}_{j,k,\eta\xi} \le \frac{\delta^n_k}{\delta^n_k + \delta^m_j} a^{m,n}_{j-1,k,\eta\xi} + \frac{\delta^m_j}{\delta^n_k + \delta^m_j} a^{m,n}_{j,k-1,\eta\xi} + \frac{\delta^m_j\delta^n_k}{\delta^n_k + \delta^m_j} h^{m,n}_{j,k,\eta\xi} \tag{5.1.3}$$

where

$$h^{m,n}_{j,k,\eta\xi} = k_{\eta\xi}\|h(t^m_j) - h(t^n_k)\|_{\eta,\xi} + \|p^m_j - p^n_k\|_{\eta,\xi}$$

Estimates (3.3.2) and (5.1.2) show that $k_{\eta\xi}$ is bounded by a number independent of n.

Let $H^{m,n}_{\eta\xi}: \Omega \to \mathrm{R}$ be defined by

$$H^{m,n}_{\eta\xi}(s, t) = b^{m,n}_{j,k,\eta\xi} \qquad \text{for} \quad (s, t) \in (t^m_{j-1}, t^m_j] \times (t^n_{k-1}, t^n_k]$$

where $b^{m,n}_{j,k,\eta\xi}$ is a solution of

$$b^{m,n}_{j,k,\eta\xi} = \frac{\delta^n_k}{\delta^n_k + \delta^m_j} b^{m,n}_{j-1,k,\eta\xi} + \frac{\delta^m_j}{\delta^n_k + \delta^m_j} b^{m,n}_{j,k-1,\eta\xi} + \frac{\delta^m_j\delta^n_k}{\delta^n_k + \delta^m_j} h^{m,n}_{j,k,\eta\xi}$$

subject to

$$b^{m,n}_{j,k,\eta\xi} = \Theta_{\delta,\eta\xi}(t^m_j - t^n_k) + \Pi^m_{\delta,\eta\xi} + \Pi^n_{\delta,\eta\xi} \qquad \text{for} \quad j = 0 \quad \text{or} \quad k = 0 \tag{5.1.4}$$

The numbers $b^{m,n}_{j,k,\eta\xi}$ form a solution of (5.1.3) with equality sign substituted for the inequality, subject to (5.1.4). By (5.1.2) and (5.1.3), we get that

$$\|\varphi^m(s) - \varphi^n(t)\|_{\eta,\xi} \leq H^{m,n}_{\eta\xi}(s, t), \qquad 0 \leq s, t \leq t, \quad m, n = 1, 2, \ldots \tag{5.1.5}$$

Define the functions $w_{\eta\xi}$, $z_{\eta\xi}$: $\Omega \to \mathbf{R}_+$ by

$$w_{\eta\xi} = k_{\eta\xi}\|h(s) - h(t)\|_{\eta,\xi} + \|p(s) - p(t)\|_{\eta,\xi}$$

$$z_{\eta\xi} = \Theta_{\delta,\eta\xi}(s - t) + \theta(t - s)\int_0^s d\tau\, w_{\eta\xi}(\tau, t - s + \tau) + \theta(s - t)\int_0^t d\tau\, w_{\eta\xi}(s - t + \tau, \tau)$$

where θ is the Heaviside function satisfying

$$\theta(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ 1 & \text{for} \quad t \geq 0 \end{cases}$$

Then, arguing as in Evans (1977), one gets

$$\lim_{m,n\to\infty} H^{m,n}_{\eta\xi}(s, t) = z_{\eta\xi}(s, t) \qquad \text{uniformly for} \quad 0 \leq s, t \leq T$$

It follows from (5.1.5) that

$$\lim \sup_{m,n\to\infty} \|\varphi^m(s) - \varphi^n(t)\|_{\eta,\xi} \leq z_{\eta\xi}(s, t)$$

As

$$z_{\eta\xi}(t, t) = \Theta_{\delta,\eta\xi}(0) = \|x_\delta - x_0\|_{\eta,\xi} < \delta$$

one concludes that

$$\lim_{n\to\infty}\|\varphi^n(t) - \varphi(t)\|_{\eta,\xi} = 0 \qquad \text{uniformly on} \quad [0, T]$$

Finally, we check that φ: $[0, T] \to \tilde{\mathscr{A}}$ is τ_w-continuous.

Let $0 \leq s < t \leq T$. Then, taking limits as $m, n \to \infty$ in (5.1.4), one gets

$$\begin{aligned}\|\varphi(s) - \varphi(t)\|_{\eta,\xi} &\leq z_{\eta\xi}(s, t)\\ &= \Theta_{\delta,\eta\xi}(t - s) + \int_0^s d\tau\, w_{\eta\xi}(\tau, t - s + \tau)\\ &= \Theta_{\delta,\eta\xi}(t - s) + \int_0^s d\tau\, (k_{\eta\xi}\|h(\tau) - h(t - s + \tau)\|_{\eta,\xi}\end{aligned}$$

$$+ \|p(\tau) - p(t - s + \tau)\|_{\eta,\xi})$$

Since $\Theta_{\delta,\eta\xi}$ is continuous on $[0, T]$ and translation is a continuous linear map on $L^1([0, T], \mathbb{D} \underline{\otimes} \mathbb{E})$, it follows that given $\epsilon > 0$ such that $0 < \delta < \epsilon$, then $\|\varphi(s) - \varphi(t)\|_{\eta,\xi} < \epsilon$ for $t - s$ sufficiently small. This shows that φ is indeed τ_w-continuous and concludes the proof. ■

Remark. For the proof of the next theorem, some other results are needed. These are obtained in the sequel.

Proposition 5.2. Let P satisfy (S_0). For almost all $t \in [0, T]$, define the multifunction $Q(t, \cdot)\colon \tilde{\mathcal{A}} \to 2^{\text{sesq}(\mathbb{D}\underline{\otimes}\mathbb{E})}$ by

$$Q(t, x)(\eta, \xi) = P(t, x)(\eta, \xi) - \langle \eta, p(t)\xi \rangle$$

$$x \in D(P(t, \cdot)) \qquad \text{and arbitrary} \quad \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$$

where p satisfies (S_2)(b). Denote the resolvent and Yosida approximation of $Q(t, \cdot)$ by $J_\lambda^Q(t, \cdot)$ and $Q_\lambda(t, \cdot)$, $\lambda > 0$, respectively. Then

(i) $J_\lambda^Q(t, x) = J_\lambda(t, x + \lambda p(t))$

(ii) $Q_\lambda(t, x) = P_\lambda(t, x + \lambda p(t)) - p(t)$

for arbitrary $x \in \tilde{\mathcal{A}}$, $\lambda > 0$, and almost all $t \in [0, T]$.

Proof. Let $x \in \tilde{\mathcal{A}}$, $\lambda > 0$, $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^\infty_{\mathbb{Y},\text{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$.

(i) Denote $J_{\lambda,\alpha\beta}(t, x + \lambda p(t))$ by y. Then

$$(\text{id}_{\tilde{\mathcal{A}}}(\cdot) + \lambda P_{\alpha\beta}(t, \cdot))^{-1}(x + \lambda p(t)) = y$$

whence

$$x + \lambda p(t) \in y + P_{\alpha\beta}(t, y)$$

whence

$$\begin{aligned} x &\in y + \lambda(P_{\alpha\beta}(t, y) - p(t)) \\ &= y + \lambda Q_{\alpha\beta}(t, y) \\ &= (\text{id}_{\tilde{\mathcal{A}}}(\cdot) + \lambda Q_{\alpha\beta}(t, \cdot))(y) \end{aligned}$$

whence

$$J_{\lambda,\alpha\beta}^Q(t, x) = y = J_{\lambda,\alpha\beta}(t, x + \lambda p(t)) \tag{5.2.1}$$

for almost all $t \in [0, T]$. This proves (i).

(ii) From (5.2.1),

$$-J_{\lambda,\alpha\beta}^Q(t, x) = -J_{\lambda,\alpha\beta}(t, x + \lambda p(t))$$

whence

$$\frac{x + \lambda p(t) - J^{Q}_{\lambda,\alpha\beta}(t, x)}{\lambda} = \frac{x + \lambda p(t) - J_{\lambda,\alpha\beta}(t, x + \lambda p(t))}{\lambda}$$

whence

$$p(t) + Q_{\lambda,\alpha\beta}(t, x) = P_{\lambda,\alpha\beta}(t, x + \lambda p(t))$$

for almost all $t \in [0, T]$. This proves (ii). ∎

Proposition 5.3. Let Q be as in Proposition 5.2. Assume that (S_0) and (S_2) hold. Then

$$\|J^{Q}_{\lambda,\alpha\beta}(t, x) - J^{Q}_{\lambda,\alpha\beta}(s, x)\|_{\eta,\xi}$$
$$\leq \lambda\|p(t) - p(s)\|_{\eta,\xi} + \lambda\|h(t) - h(s)\|_{\eta,\xi}L'(\|x\|_{\eta,\xi})(1 + \|Q_{\lambda,\alpha\beta}(t, x)\|_{\eta,\xi})$$

for almost all $s, t \in [0, T]$, arbitrary $\lambda \in (0, 1]$, and $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^{\infty}_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$, where L' is some continuous, positive, nondecreasing function on $\mathbb{R}_+$.

Proof. The estimate is obtained as follows:

$$\|J^{Q}_{\lambda,\alpha\beta}(t, x) - J^{Q}_{\lambda,\alpha\beta}(s, x)\|_{\eta,\xi}$$
$$= \|J_{\lambda,\alpha\beta}(t, x + \lambda p(t)) - J_{\lambda,\alpha\beta}(s, x + \lambda p(s))\|_{\eta,\xi}$$

[by (i) of Proposition 5.2]

$$\leq \lambda\|p(t) - p(s)\|_{\eta,\xi}$$
$$+ \|J_{\lambda,\alpha\beta}(t, x + \lambda p(t)) - J_{\lambda,\alpha\beta}(s, x + \lambda p(t))\|_{\eta,\xi}$$
$$= \lambda\|p(t) - p(s)\|_{\eta,\xi}$$
$$+ \lambda\|P_{\lambda,\alpha\beta}(t, x + \lambda p(t)) - P_{\lambda,\alpha\beta}(s, x + \lambda p(t))\|_{\eta,\xi}$$
$$\leq \lambda\|p(t) - p(s)\|_{\eta,\xi}$$
$$+ \lambda\|h(t) - h(s)\|_{\eta,\xi}L(\|x + \lambda p(t)\|_{\eta,\xi})[1 + \|P_{\lambda,\alpha\beta}(t, x + \lambda p(t))\|_{\eta,\xi}]$$

[by (S_2)]

$$\leq \lambda\|p(t) - p(s)\|_{\eta,\xi} + \lambda\|h(t) - h(s)\|_{\eta,\xi}L'(\|x\|_{\eta,\xi})(1 + \|Q_{\lambda,\alpha\beta}(t, x)\|_{\eta,\xi})$$

by using (ii) of Proposition 5.2, with $L'(\sigma) = L(\sigma + c^1_{\eta\xi})$, for some positive constant $c^1_{\eta\xi}$, by the hypothesis on p. This concludes the proof. ∎

Remark. From Proposition 5.3, one gets

$$\|Q_{\lambda,\alpha\beta}(t, x) - Q_{\lambda,\alpha\beta}(s, x)\|_{\eta,\xi}$$
$$\leq \|p(t) - p(s)\|_{\eta,\xi} + \|h(t) - h(s)\|_{\eta,\xi} L'(\|x\|_{\eta,\xi})(1 + \|Q_{\lambda,\alpha\beta}(t, x)\|_{\eta,\xi})$$

whence

$$|Q(s, x)|_{\eta\xi} \leq |Q(t, x)|_{\eta\xi} + \|p(t) - p(s)\|_{\eta,\xi} + \|h(t) - h(s)\|_{\eta,\xi} L'(\|x\|_{\eta,\xi})(1 + |Q(t, x)|_{\eta\xi}) \quad (5.3.1)$$

showing that $D(Q(t, \cdot))$, $\hat{D}(Q(t, \cdot))$, and $\overline{D(Q(t, \cdot))}$ are constant almost everywhere.

Proposition 5.4. The inclusion

$$\frac{\langle \eta, (x_k^n - x_{k-1}^n)\xi \rangle}{t_k^n - t_{k-1}^n} \in -Q(t_k^n, x_k^n)(\eta, \xi)$$

implies

$$Q_{\delta_{k-1}^n}(t_{k-1}^n, x_{k-2}^n)(\eta, \xi) \in Q(t_{k-1}^n, x_{k-1}^n)(\eta, \xi)$$

for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^\infty_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$.

Proof. With $\delta_k^n = t_k^n - t_{k-1}^n$, replacing k by $k - 1$ in the given inclusion, one gets

$$-\langle \eta, x_{k-2}^n \xi \rangle \in -\langle \eta, x_{k-1}^n \xi \rangle - \delta_{k-1}^n Q(t_{k-1}^n, x_{k-1}^n)(\eta, \xi)$$
$$\Rightarrow -x_{k-2}^n \in -x_{k-1}^n - \delta_{k-1}^n Q_{\alpha\beta}(t_{k-1}^n, x_{k-1}^n)$$
$$\Rightarrow x_{k-2}^n \in x_{k-1}^n + \delta_{k-1}^n Q_{\alpha\beta}(t_{k-1}^n, x_{k-1}^n)$$
$$\Rightarrow Q_{\delta_{k-1}^n,\alpha\beta}(t_{k-1}^n, x_{k-2}^n) = x_{k-1}^n$$
$$\Rightarrow Q(t_{l-1}^n, Q_{\delta_{k-1}^n,\alpha\beta}(t_{k-1}^n, x_{k-2}^n)) = Q(t_{k-1}^n, x_{k-1}^n)$$

As

$$Q_{\delta_{k-1}^n,\alpha\beta}(t_{k-1}^n, x_{k-2}^n) \in Q(t_{l-1}^n, Q_{\delta_{k-1}^n,\alpha\beta}(t_{k-1}^n, x_{k-2}^n))$$

by part 1(iii), of Theorem 4.1 in Ekhaguere (1995), one gets

$$Q_{\delta_{k-1}^n,\alpha\beta}(t_{k-1}^n, x_{k-2}^n) \in Q(t_{l-1}^n, x_{k-1}^n)$$

whence

$$Q_{\delta_{k-1}^n}(t_{k-1}^n, x_{k-2}^n)(\eta, \xi) \in Q(t_{l-1}^n, x_{k-1}^n)(\eta, \xi)$$

for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^\infty_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$. This proves the proposition. ■

Proposition 5.5. Adopt the notation of Theorem 5.1. Denote $\|p_k^n + (x_{k-1}^n - x_k^n)/\delta_k^n\|_{\eta,\xi}$ by $c_{\eta\xi,kn}$, for $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^\infty_{\Upsilon,\mathrm{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$. Then, $c_{\eta\xi,kn}$ is bounded by a number independent of k and n.

Proof. One has

$$\begin{aligned} c_{\eta\xi,kn} &= \left\| p_k^n + \frac{x_{k-1}^n - x_k^n}{\delta_k^n} \right\|_{\eta,\xi} \\ &= \|P_{\delta_k^n,\alpha\beta}(t_k^n, x_{k-1}^n + \delta_k^n p_k^n)\|_{\eta,\xi} \\ &\quad \text{[compare (4.1.3)]} \\ &\le \|P_{\delta_k^n,\alpha\beta}(t_k^n, x_{k-1}^n + \delta_k^n p_k^n) - P_{\delta_k^n,\alpha\beta}(t_k^n, x_{k-1}^n)\|_{\eta,\xi} \\ &\quad + \|P_{\delta_k^n,\alpha\beta}(t_k^n, x_{k-1}^n)\|_{\eta,\xi} \\ &\le \|p_k^n\|_{\eta,\xi} + \|P_{\delta_k^n,\alpha\beta}(t_k^n, x_{k-1}^n)\|_{\eta,\xi} \\ &\le \|p_k^n\|_{\eta,\xi} + |P(t_k^n, x_{k-1}^n)|_{\eta\xi} \end{aligned}$$

The map p is τ_w-essentially bounded, since it is of essentially bounded variation, by hypothesis. Hence, there is a number $c^1_{\eta\xi}$ such that

$$\|p_k^n\|_{\eta,\xi} = \|p(t_k^n)\|_{\eta,\xi} \le \operatorname{ess}\sup_{0\le t\le T} \|p(t)\|_{\eta,\xi} \le c^1_{\eta\xi}$$

To estimate the quantity $|P(t_k^n, x_{k-1}^n)|_{\eta\xi}$, let the multifunction Q be defined as in Proposition 5.2. Then, the inclusion

$$\frac{\langle \eta, (x_k^n - x_{k-1}^n)\xi\rangle}{t_k^n - t_{k-1}^n} \in -P(t_k^n, x_k^n)(\eta, \xi) + \langle \eta, p_k^n \xi\rangle, \qquad k = 1, 2, \ldots, N(n)$$

may be rewritten as follows:

$$\frac{\langle \eta, (x_k^n - x_{k-1}^n)\xi\rangle}{t_k^n - t_{k-1}^n} \in -Q(t_k^n, x_k^n)(\eta, \xi), \qquad k = 1, 2, \ldots, N(n)$$

Set

$$q_{k,\eta\xi} = |Q(t_k^n, x_{k-1}^n)|_{\eta\xi}, \qquad k = 1, 2, \ldots, N(n)$$

Employing (5.3.1), with $s = t_k^n$, $t = t_{k-1}^n$, and $x = x_{k-1}^n$, one gets

$$\begin{aligned} q_{k,\eta\xi} &\le |Q(t_{k-1}^n, x_{k-1}^n)|_{\eta\xi} + \|p^n(t_{k-1}^n) - p^n(t_k^n)\|_{\eta,\xi} \\ &\quad + \|h(t_{k-1}^n) - h(t_k^n)\|_{\eta,\xi} L'(\|x\|_{\eta,\xi})(1 + |Q(t_{k-1}^n, x_{k-1}^n)|_{\eta\xi}) \qquad (5.5.1) \end{aligned}$$

By Proposition 5.4,

$$Q_{\delta^n_{k-1}}(t^n_{k-1}, x^n_{k-2})(\eta, \xi) \in Q(t^n_{l-1}, x^n_{k-1})(\eta, \xi)$$

showing that

$$\begin{aligned} |Q(t^n_{l-1}, x^n_{k-1})|_{\eta\xi} &\leq |Q_{\delta^n_{k-1}}(t^n_{k-1}, x^n_{k-2})(\eta, \xi)| \\ &\leq |Q(t^n_{k-1}, x^n_{k-2})|_{\eta\xi} \\ &= q_{k-1,\eta\xi} \end{aligned}$$

Using this result in (5.5.1), one gets

$$q_{k,\eta\xi} \leq q_{k-1,\eta\xi} + b_{k,\eta\xi}(1 + q_{k-1,\eta\xi}) = d_{k,\eta\xi}q_{k-1,\eta\xi} + b_{k,\eta\xi}$$

where

$$\begin{aligned} b_{k,\eta\xi} &= \|p^n(t^n_{k-1}) - p^n(t^n_k)\|_{\eta,\xi} \\ &\quad + \|h(t^n_{k-1}) - h(t^n_k)\|_{\eta,\xi}L'(\|x\|_{\eta,\xi}) \\ d_{k,\eta\xi} &= 1 + b_{k,\eta\xi} \end{aligned}$$

From the last inequality, it follows that

$$q_{k,\eta\xi} \leq \left(a_{1,\eta\xi} + \sum_{j=2}^{k} b_{j,\eta\xi}\right) \exp\left(\sum_{j=2}^{k} b_{j,\eta\xi}\right)$$

By the definition of p^n and h^n,

$$\begin{aligned} \sum_{j=2}^{k} b_{j,\eta\xi} &\leq \mathrm{Var}_{\eta\xi}(p^n) + L'(\|x\|_{\eta,\xi}) \,\mathrm{Var}_{\xi\eta}(h^n) \\ &\leq \mathrm{Var}_{\eta\xi}(p) + L'(\|x\|_{\eta,\xi}) \,\mathrm{Var}_{\eta\xi}(h) \end{aligned}$$

Here

$$\mathrm{Var}_{\eta\xi}(z) = \sup\left(\sum_{j=1}^{n} |z(t_j)(\eta, \xi) - z(t_{j-1})(\eta, \xi)|\right)$$

for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, where the supremum is taken over all partitions $\{t_j\}_{j=0}^{n}$ of $[0, T]$ lying outside the subset of $[0, T]$ on which the map z of essentially bounded variation is undefined.

Since $x_0 \in \hat{D}$, using (5.5.1) and the fact that both p and h are τ_w-essentially bounded, one shows that $q_{1,\eta\xi} = |Q(t^n_1, x^n_0)|_{\eta\xi}$ is bounded by a number independent of n. From this, one gets that $q_{k,\eta\xi} = |Q(t_k^n, x_{k-1}^n)|_{\eta\xi}$ is bounded by a number which does not depend on k and n. As

$$\begin{aligned}
&|P(t_k^n, x_{k-1}^n)|_{\eta\xi} \\
&\quad = \lim_{\lambda \searrow 0} |P_\lambda(t_k^n, x_{k-1}^n)(\eta, \xi)| \\
&\quad = \lim_{\lambda \searrow 0} |Q_\lambda(t_k^n, x_{k-1}^n - \lambda p(t_k^n))(\eta, \xi) + \langle \eta, p(t_k^n)\xi\rangle| \\
&\quad \leq \lim_{\lambda \searrow 0} |Q_\lambda(t_k^n, x_{k-1}^n - \lambda p(t_k^n))(\eta, \xi) - Q_\lambda(t_k^n, x_{k-1}^n)(\eta, \xi)| \\
&\qquad + \lim_{\lambda \searrow 0} |Q_\lambda(t_k^n, x_{k-1}^n)(\eta, \xi)| + \|p(t_k^n)\|_{\eta,\xi} \\
&\quad \leq |Q(t_k^n, x_{k-1}^n)|_{\eta\xi}| + 2\|p(t_k^n)\|_{\eta,\xi}
\end{aligned}$$

it follows that $|P(t_k^n, x_{k-1}^n)|_{\eta\xi}$ is also bounded by a number that is independent of both k and n. This concludes the proof. ∎

Theorem 5.6. Let $\overline{T} > 0$, $x_0 \in \hat{D}$. Assume that (S_0) and (S_2) hold. Then the conclusions of Theorem 5.1 remain true.

Proof. It is easy to show as in Theorem 5.1 that claims (i)–(vi) are again valid. In proving (vii), one notes that the estimates obtained in the proof of Theorem 5.1 also remain valid. But now the constant $k_{\eta\xi}$ depends on both

$$l_{\eta\xi}^1 = \max_{0 \leq k \leq n} (\|x\|_{\eta,\xi}) \qquad \text{and} \qquad l_{\eta\xi}^2 = \left\| p_k^n + \frac{x_{k-1}^n - x_k^n}{t_k^n - t_{k-1}^n} \right\|_{\eta,\xi}$$

in view of Proposition 4.1. It must be shown that $k_{\eta\xi}$ has a bound that is independent of both k and n. We have already seen that $l_{\eta\xi}^1 = \max_{0\leq k\leq n}(\|x\|_{\eta,\xi})$ has such a bound. By Proposition 5.5, $l_{\eta\xi}^2$ also has a bound that does not depend on both k and n. The assertion (vii) now follows as in Theorem 5.1. This concludes the proof. ∎

6. COMPARISON OF LIMIT SOLUTIONS

Let $T > 0$. Consider the initial value stochastic differential inclusions

$$\begin{aligned}
dX(t) \in &-(E(t, X(t))\, d\Lambda_\pi(t) + F(t, X(t))\, dA_f(t) + G(t, X(t))\, dA_g^+(t) \\
&+ H(t, X(t))\, dt) + p(t)\, dt \qquad \text{almost all} \quad t \in (0, T] \qquad (3.2.1)_X \\
X(0) = x_0 &\qquad \text{for some} \quad x_0 \in \tilde{\mathcal{A}}
\end{aligned}$$

and

$$dY(t) \in -(E(t, Y(t))\, d\Lambda_\pi(t) + F(t, Y(t))\, dA_f(t) + G(t, Y(t))\, dA_g^+(t)$$
$$+ H(t, Y(t))\, dt) + q(t)\, dt \qquad \text{almost all} \quad t \in (0, T]$$
$$Y(0) = y_0 \qquad \text{for some} \quad y_0 \in \tilde{\mathcal{A}} \tag{3.2.1$_Y$}$$

By Theorem 6.2 of Ekhaguere (1992), these are equivalent to the differential inclusions

$$\frac{d}{dt}\langle \eta, X(t)\xi\rangle \in -P(t, X(t))(\eta, \xi) + \langle \eta, p(t)\xi\rangle$$
$$\text{for almost all} \quad t \in (0, T] \tag{3.2.1$_P$}$$
$$X(0) = x_0$$
$$\frac{d}{dt}\langle \eta, Y(t)\xi\rangle \in -P(t, Y(t))(\eta, \xi) + \langle \eta, q(t)\xi\rangle$$
$$\text{for almost all} \quad t \in (0, T] \tag{3.2.1$'_P$}$$
$$Y(0) = y_0$$

with P as defined in (3.2.2) and $p, q\colon [0, T] \to \tilde{\mathcal{A}}$.

In this section, we compare the limit solutions of $(3.2.1)_P$ and $(3.2.1)'_P$, which are constructed as in Theorems 5.1 and 5.6.

Theorem 6.1. Let $\overline{T} > 0$ and (S_0). Suppose that both $(3.2.1)_P$ and $(3.2.1)'_P$ satisfy the hypotheses of Theorem 5.1 or else the hypotheses of Theorem 5.6. Then the limit solutions φ_∞ of $(3.2.1)_P$ and ϕ_∞ of $(3.2.1)'_P$, which exist by (vii) of Theorem 5.1 or Theorem 5.6, satisfy the integral inequality

$$\|\varphi_\infty(t) - \phi_\infty(t)\|_{\eta,\xi} \leq \|\varphi_\infty(s) - \phi_\infty(s)\|_{\eta,\xi}$$
$$+ \int_s^t d\tau\, [p(\tau) - q(\tau), \varphi_\infty(\tau) - \phi_\infty(\tau)]_{\eta\xi}$$

for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ and almost all $0 \leq s < t \leq T$.

Proof. By either Theorem 5.1 or Theorem 5.6, there are partitions

$$\Delta_m = \{0 = s_0^m < s_1^m < \cdots < s_{M(m)}^m \equiv S(m)\}$$
$$\Delta'_n = \{0 = t_0^n < t_1^n < \cdots < t_{N(n)}^n \equiv T(n)\}$$

and sequences $\{x_j^m\}$, $\{y_k^n\}$, $\{p_j^m\}$, $\{q_k^n\}$ of members of $\tilde{\mathscr{A}}$ such that

$$\frac{\langle \eta, (x_j^m - x_{j-1}^m)\xi\rangle}{s_j - s_{j-1}} \in -P(s_j, x_j^m)(\eta, \xi) + \langle \eta, p_j^m \xi\rangle,$$

$$j = 1, 2, \ldots, M(m); \quad m = 1, 2, \ldots$$

and

$$\frac{\langle \eta, (y_k^n - y_{k-1}^n)\xi\rangle}{t_k - t_{k-1}} \in -P(t_k, y_k^n)(\eta, \xi) + \langle \eta, q_k^n \xi\rangle,$$

$$k = 1, 2, \ldots, N(n); \quad n = 1, 2, \ldots$$

$\eta, \xi \in \mathrm{D} \underline{\otimes} \mathrm{E}$.

Set $\gamma_j^m = s_j^m - s_{j-1}^m$, $\delta_k^n = t_k^n - t_{k-1}^n$, $x_0^m \equiv x_0$, and $y_0^n \equiv y_0$. Define the $\tilde{\mathscr{A}}$-valued maps φ^m, p^m, ϕ^n, q^n, h^m, $h^{(n)}$ by

$$\varphi^m(s) = x_j^m, \qquad p^m(s) = p_j^m, \qquad h^m(s) = h(s_j^m) \qquad \text{for} \quad s \in (s_{j-1}^m, s_j^m]$$

and

$$\phi^n(t) = y_k^n, \qquad q^n(t) = q_k^n, \qquad h^{(n)}(t) = h(t_k^n) \qquad \text{for} \quad t \in (t_{k-1}^n, t_k^n]$$

Choose Δ_m and Δ_n' such that

$$\lim_{m\to\infty} \|p^m - p\|_{1,\eta\xi}^{(m)} = 0 = \lim_{n\to\infty} \|q^n - q\|_{1,\eta\xi}^{(n)}$$

and

$$\lim_{n\to\infty} \|h^m - h\|_{1,\eta\xi}^{(m)} = 0 = \lim_{n\to\infty} \|h^{(n)} - h\|_{1,\eta\xi}^{(n)}$$

$\eta, \xi \in \mathrm{D} \underline{\otimes} \mathrm{E}$, where $\{\|\cdot\|_{1,\eta\xi}^{(m)}: \eta, \xi \in \mathrm{D} \underline{\otimes} \mathrm{E}\}$ and $\{\|\cdot\|_{1,\eta\xi}^{(n)}: \eta, \xi \in \mathrm{D} \underline{\otimes} \mathrm{E}\}$ are the families of seminorms of $L^1([0, S(m)], \mathrm{D} \underline{\otimes} \mathrm{E})$ and $L^1([0, T(n)], \mathrm{D} \underline{\otimes} \mathrm{E})$, respectively.

Introduce

$$\Delta_{m,n} = \{(s_{j-1}^m, s_j^m] \times (t_{k-1}^n, t_k^n]: j = 1, 2, \ldots, M(m); k = 1, 2, \ldots, N(n)\}$$

$$\Omega_{m,n} = (0, M(m)] \times (0, N(n)], \qquad \Omega = (0, S] \times (0, T]$$

Pick any $\lambda > 0$. Then, by Proposition (4.3), the numbers $a_{jk,\eta\xi}^{m,n} = \|x_j^m - y_k^n\|_{\eta,\xi}$ satisfy the inequality (5.1.3) with

$$h_{jk,\eta\xi}^{m,n} = k_{\eta\xi}\|h(s_j^m) - h(t_k^n)\|_{\eta,\xi} + [x_j^m - y_k^n, p_j^m - q_k^n]_{\lambda,\eta\xi}$$

and δ_j^m replaced by γ_j^m. As in the proof of Theorem 5.1, one gets

$$\|x_j^m - x_0^m\|_{\eta,\xi} \leq \Theta_{\delta,\eta\xi}^1(s_j^m) + \Pi_{\delta,\eta\xi}^{1,m}$$

and

$$\|y_k^n - y_0^n\|_{\eta,\xi} \leq \Theta^2_{\delta,\eta\xi}(t_k^n) + \Pi^{2,n}_{\delta,\eta\xi}$$

with the obvious definitions of $\Theta^j_{\delta,\eta\xi}$, $\Pi^j_{\delta,\eta\xi}$, $j = 1, 2$. The functions $\Theta^j_{\delta,\eta\xi}$ are continuous, with $0 \leq \Theta^j_{\delta,\eta\xi}(t) \leq \delta$, $j = 1, 2$, and $\Pi^{1,m}_{\delta,\eta\xi} \to 0$, $\Pi^{2,n}_{\delta,\eta\xi} \to 0$, as $m, n \to \infty$.

Define $\Pi^n_{\psi,\eta\xi} = \max(\Pi^{1,n}_{\delta,\eta\xi}, \Pi^{2,n}_{\delta,\eta\xi})$ and let $\Theta_{\delta,\eta\xi}$ be the even extension to $[-\overline{T}, \overline{T}]$ of the function $t \mapsto \max(\Theta^1_{\delta,\eta\xi}(t), \Theta^2_{\delta,\eta\xi}(t))$, $t \in [0, \overline{T}]$. As in the proof of Theorem 5.1, introduce the maps $w_{\eta\xi}$, $z_{\eta\xi}$, and $H^{m,n}_{\eta\xi}$ from $\Omega \to \mathbb{R}$ by

$$w_{\eta\xi}(s, t) = k_{\eta\xi}\|h(s) - h(t)\|_{\eta,\xi} + [p(s) - q(t), \varphi_\infty(s) - \phi_\infty(t)]_{\lambda,\eta\xi}$$

$$z_{\eta\xi}(s, t) = \Theta_{\delta,\eta\xi}(t - s) + \|x_0 - y_0\|_{\eta,\xi} + \theta(t - s) \int_0^s d\tau\, w_{\eta\xi}(\tau, t - s + \tau)$$

$$+ \theta(s - t) \int_0^t d\tau\, w_{\eta\xi}(s - t + \tau, \tau)$$

where θ is the Heaviside function, and

$$H^{m,n}_{\eta\xi}(s, t) = b^{m,n}_{j,k,\eta\xi}, \qquad (s, t) \in (s^m_{j-1}, s^m_j] \times (t^n_{k-1}, t^n_k]$$

where the numbers $b^{m,n}_{j,k,\eta\xi}$ solve the iterative scheme

$$b^{m,n}_{j,k,\eta\xi} = \frac{\delta^n_k}{\delta^n_k + \gamma^m_j} \cdot b^{m,n}_{j-1,k,\eta\xi} + \frac{\gamma^m_j}{\delta^n_k + \gamma^m_j} \cdot b^{m,n}_{j,k-1,\eta\xi} + \frac{\delta^n_k \gamma^m_j}{\delta^n_k + \gamma^m_j} \cdot h^{m,n}_{j,k,\eta\xi}$$

$$b^{m,n}_{j,k,\eta\xi} = \Theta_{\delta,\eta\xi}(s^m_j - t^n_k) + \Pi^n_{\delta,\eta\xi} + \Pi^m_{\delta,\eta\xi} + \|x_0 - y_0\|_{\eta,\xi} \qquad \text{for} \quad j = 0 \text{ or } k = 0$$

Following the same arguments as in the proof of Theorem 5.1, one gets

$$\|\varphi^m(s) - \phi^n(t)\|_{\eta,\xi} \leq H^{m,n}_{\eta\xi}(s, t)$$
$$\lim_{m,n\to\infty} H^{m,n}_{\eta\xi}(s, t) = z_{\eta\xi}(s, t)$$

and

$$\|\varphi_\infty(s) - \phi_\infty(t)\|_{\eta,\xi} \leq z_{\eta\xi}(s, t), \qquad (s, t) \in \Omega$$

Hence

$$\|\varphi_\infty(s) - \phi_\infty(t)\|_{\eta,\xi}$$
$$\leq z_{\eta\xi}(s, t)$$
$$\leq \delta + \|x_0 - y_0\|_{\eta,\xi} + \int_0^t d\tau\, [p(\tau) - q(\tau), \varphi_\infty(\tau) - \phi_\infty(\tau)]_{\eta\xi}$$

by letting $\delta \searrow 0$ and $\lambda \searrow 0$. This concludes the proof. ■

Remark. Using (vi) of Proposition 2.1, we obtain from the last inequality

$$\|\varphi_\infty(s) - \phi_\infty(t)\|_{\eta,\xi} \leq \delta + \|x_0 - y_0\|_{\eta,\xi} + \int_0^t d\tau \, \|p(\tau) - q(\tau)\|_{\eta,\xi} \tag{6.1.1}$$

7. THE EVOLUTION OPERATOR

Let $s > 0$ be fixed. Consider again the initial value stochastic differential inclusion

$$\begin{aligned} dX(t) \in & -(E(t, X(t))\, d\Lambda_\pi(t) + F(t, X(t))\, dA_f(t) + G(t, X(t))\, dA_g^+(t) \\ & + H(t, X(t))\, dt) + p(t)\, dt \qquad \text{almost all} \quad t \in (s, T] \qquad (3.2.1)_X \\ X(s) = x_s & \qquad \text{for some} \quad x_s \in \tilde{\mathcal{A}} \end{aligned}$$

which is equivalent to the differential inclusion

$$\begin{aligned} \frac{d}{dt}\langle \eta, X(t)\xi \rangle & \in -P(t, X(t))(\eta, \xi) + \langle \eta, p(t)\xi \rangle \\ & \qquad \text{for almost all} \quad t \in (s, T] \qquad (3.2.1)_P \\ X(s) & = x_s \end{aligned}$$

for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^\infty_{\Upsilon,\mathrm{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$.

In Ekhaguere (1995), the case corresponding to $p \equiv 0$ was considered and it was shown that Problem $(3.2.1)_X$ has a unique adapted solution. This same conclusion applies also to any nonzero $p \in C([s, T], \mathbb{D} \underline{\otimes} \mathbb{E})$. In this paper, we have been examining a more general setting.

Suppose that Problem $(3.2.1)_X$ has a unique adapted solution φ. One may interpret $(3.2.1)_X$ as describing a system whose state at time s is $\varphi(s) = x_s$, while $\varphi(t)$ is the state of the system at some later time $t \geq s$. One says that the system has evolved from the state $\varphi(s)$ to the state $\varphi(t)$, $t \geq s$. This transition may be described by means of a transformation $U(t, s)$ which moves $\varphi(s)$ to $\varphi(t)$ thus:

$$U(t, s)\varphi(s) = \varphi(t), \qquad t \geq s \tag{7.1}$$

It follows that

$$U(s, s)\varphi(s) = \varphi(s) \tag{7.2a}$$

and by the assumed uniqueness of the solution of Problem $(3.2.1)_X$,

$$U(t, r)U(r, s)\varphi(s) = U(t, s)\varphi(s) \tag{7.2b}$$

for $s \leq r \leq t$. The relations (7.2a) and (7.2b) are called *evolution conditions.*

Definition. A map U from the set $\{(t, s) \in \mathbb{R}_+^2 : 0 \leq s \leq t \leq T\}$ to the set of all operators on $\tilde{\mathcal{A}}$ is called an *evolution operator* if it satisfies (7.2a) and (7.2b).

Remark. The *evolution operators* described in this paper are, in general, nonlinear.

Definition. Let the hypotheses of Theorem 5.1 or Theorem 5.6 hold and let φ_∞ be the limit solution whose existence is affirmed by these results with the initial time 0 replaced by $s \geq 0$. Define

$$U(t, s)x_s \equiv \varphi_\infty(t), \qquad t \in [s, T] \tag{7.2c}$$

By Theorem 6.1, φ_∞ is uniquely determined. Hence, $U(t, s)$ is well defined.

Proposition 7.1. Let $s \in (0, T]$. Assume that $x_0, y_0 \in \tilde{\mathcal{A}}$ and the hypotheses of Theorems 5.1 and 5.6 hold on $[s, T]$ for both x_0 and y_0. Then

$$\|U(t, s)x_0 - U(t, s)y_0\|_{\eta,\xi} \leq \|x_0 - y_0\|_{\eta,\xi} \tag{7.3}$$

for all $s \leq t \leq T$, $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$.

Proof. This follows from (6.1.1).

Remark. One concludes from (7.3) that $U(t, s)$ admits a unique extension from $\hat{D}$ to $\overline{D}$ under the hypotheses of Theorem 5.6.

Proposition 7.2. If φ_∞ is the limit solution arising from the hypotheses of Theorem 5.6, then $\varphi_\infty(t)$ lies in $\hat{D}$ for every $t \in [0, T]$.

Proof. Let $t \in [0, T]$ be such that (S_0) and (S_2) hold. Construct the partitions $\{\Delta_n\}_{n\in\mathbb{N}}$ and the sets $\{p_k^n\}_{k=1}^n$, $\{x_k^n\}_{k=1}^n$ as in Theorem 5.6. As demonstrated in the proof of Proposition 5.5, $|P(t_k^n, x_{k-1}^n)(\eta, \xi)|$ is bounded by a constant independent of k and n, for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$. Since $x_{k-1}^n \in D(P(t_k^n, \cdot)) = \hat{D}$, by (S_2)(a),

$$\begin{aligned} &|P(t, x_{k-1}^n)(\eta, \xi)| \\ &\leq |P(t_k^n, x_{k-1}^n)(\eta, \xi)| \\ &\quad + \|h(t_k^n) - h(t)\|_{\eta,\xi} L(\|x_{k-1}^n\|_{\eta,\xi})(1 + |P(t_k^n, x_{k-1}^n)(\eta, \xi)|) \end{aligned}$$

showing that the right-hand side of this inequality is bounded by a constant independent of k and n. Choosing $k = k(n)$ such that $t_k^n \to \infty$ as $n \to \infty$, then x_{k-1}^n τ_w-converges to $\varphi_\infty(t)$. As

$$
\begin{aligned}
&|P_\lambda(t, \varphi_\infty(t))(\eta, \xi)| \\
&\quad \le |P_\lambda(t, x_{k-1}^n)(\eta, \xi)| + |P_\lambda(t, x_{k-1}^n)(\eta, \xi) - P_\lambda(t, \varphi_\infty(t))(\eta, \xi)| \\
&\quad \le |P(t, x_{k-1}^n)|_{\eta\xi} + \frac{1}{\lambda} \|x_{k-1}^n - \varphi_\infty(t)\|_{\eta,\xi}
\end{aligned}
$$

and $\{|P(t, x_{k-1}^n)|_{\eta\xi}\}$ is bounded by a constant independent of k and n, it follows that $\varphi_\infty(t) \in \hat{D}$.

Finally, let t in $[0, T]$ be arbitrary and $\{t_n\} \subset [0, T]$ be such that $\varphi_\infty(t_n) \in \hat{D}$ and $t_n \to t$. Then, choosing $t_0 \in [0, T]$ such that $P(t_0, \cdot)$ is hypermaximal monotone with $\hat{D}(P(t_0, \cdot)) = \hat{D}$, the numbers $|P(t_0, \varphi_\infty(t_n))(\eta, \xi)|$ are bounded by a constant independent of n and $\varphi_\infty(t_n)$ τ_w-converges to $\varphi_\infty(t)$. Hence, the foregoing inequality also gives $\varphi_\infty(t) \in \hat{D}$. This concludes the proof. ∎

Remark. The next result demonstrates that the operator U defined in (7.2c) satisfies the evolution conditions (7.2a) and (7.2b).

Proposition 7.3. Let $0 \le r < s < t < T$. Suppose that the hypotheses of Theorem 5.1 (resp. Theorem 5.6) hold for Problem $(3.2.1)_X$ on $[r, T]$ for some initial value $X(r) = x_0$. Then:

(i) The hypotheses of Theorem 5.1 (resp. Theorem 5.6) also hold on $[s, T]$ with initial data $y_0 = U(s, r)x_0$.

(ii) $U(t, r)x_0 = U(t, s)U(s, r)x_0$.

Proof. (i) Observe that $y_0 \in \overline{D}$, since $U(s, r)x_0$ is the τ_w-limit of solutions $x_k^n \in \overline{D}$ of approximation schemes for Problem $(3.2.1)_X$. Hence, Theorem 5.1 holds on $[s, T]$ whenever it holds on $[r, T]$. By Proposition 7.2, $y_0 \in \hat{D}$ if Theorem 5.6 holds.

(ii) This is proved by comparing, as in Sections 5 and 6, the two stochastic processes $U(\cdot, r)x_0$ and $U(\cdot, s)U(s, r)x_0$ by constructing two approximating schemes that converge to these processes, whence one finds that $U(\cdot, r)x_0$ and $U(\cdot, s)U(s, r)x_0$, which is equal to $U(\cdot, s)y_0$, coincide. This concludes the proof. ∎

Definition. The family of multifunctions $\{P(t, \cdot): t \in [0, T]\}$ is called the *generator* of the evolution operator $U(\cdot, \cdot)$ described in Proposition 7.3.

8. LIMIT SOLUTIONS AND SOLUTIONS

In this section, we describe the relationship between a limit solution constructed in Theorem 5.1 or Theorem 5.6 and a solution of Problem $(3.2.1)_X$.

Remark. We give two results in this section, the first of which establishes that a solution of Problem $(3.2.1)_X$ can be constructed as the limit of solutions of approximation schemes.

Theorem 8.1. Let $\overline{T} > 0$, $P \in \text{Hypmax}([0, \overline{T}] \times \tilde{\mathcal{A}})$, $p \in L^1([0, \overline{T}], \mathbb{D} \underline{\otimes} \mathbb{E})$, and φ a solution of Problem $(3.2.1)_X$ such that the sesq($\mathbb{D} \underline{\otimes} \mathbb{E}$)-valued map $(\eta, \xi) \mapsto (d/dt)\langle \eta, \varphi(\cdot)\xi\rangle$, $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, on $[0, T]$ is in $L^1([0, T], \mathbb{D} \underline{\otimes} \mathbb{E})$. Suppose that P also satisfies either (S_1) or (S_2). Then, for every $0 < s < T < \overline{T}$, φ is the τ_{con}-limit of solutions of approximating schemes on $[s, T]$, in the sense that there are partitions $\{\Delta_n\}_{n \in \mathbb{N}}$, with $\Delta_n = \{s = t_0^n < t_1^n < \cdots < t_{N(n)}^n \equiv T(n)\}$, of $[s, T(n)]$ and sequences $\{x_k^n\}_{k=0}^{N(n)}$, $\{p_k^n\}_{k=1}^{N(n)}$, such that:

(i) $T \leq T(n) \leq \overline{T}$.

(ii) $\lim_{n\to\infty} \max_{1\leq k\leq N(n)} (t_k^n - t_{k-1}^n) = 0$.

(iii) $x_0^n = x_0$.

(iv) The step functions p^n, defined by

$$p^n(t) = p_k^n \qquad \text{for} \quad t \in (t_{k-1}^n, t_k^n]$$

satisfy $\lim_{n\to\infty}\|p^n - p\|_{1,\eta\xi}^{(n)} = 0$, for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$.

(v) The step functions φ^n, defined by

$$\varphi^n(t) = x_k^n \qquad \text{for} \quad t \in (t_{k-1}^n, t_k^n]$$

solve

$$\frac{\langle \eta, (x_k^n - x_{k-1}^n)\xi\rangle}{t_k^n - t_{k-1}^n} \in -P(t_k^n, x_k^n)(\eta, \xi) + \langle \eta, p_k^n \xi\rangle$$

$k = 1, 2, \ldots, N(n)$; $n = 1, 2, \ldots$; $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$.

(vi) The step functions h^n, defined by

$$h^n(t) = h(t_k^n) \qquad \text{for} \quad t \in (t_{k-1}^n, t_k^n]$$

satisfy $\lim_{n\to\infty}\|h^n - h\|_{1,\eta\xi}^{(n)} = 0$ for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$.

(vii) φ^n converges to φ in $C([s, T], \mathbb{D} \underline{\otimes} \mathbb{E})$.

Here $\{\|\cdot\|_{1,\eta\xi}^{(n)}: \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$ is the family of seminorms of $L^1([0, T(n)], \mathbb{D} \underline{\otimes} \mathbb{E})$.

Proof. By Proposition 2.3, there are partitions $\{\Delta_n\}_{n \in \mathbb{N}}$, with $\Delta_n = \{s = t_0^n < t_1^n < \cdots < t_{N(n)}^n \equiv T(n)\}$, of $[s, T(n)]$ such that: (i) and (ii) are valid; the complex-valued function $t \mapsto \langle \eta, \varphi(t)\xi\rangle$, $t \in [s, T]$, is differentiable at each partition point t_k^n, except perhaps at s, for all $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$; the inclusion

$$\left(\frac{d}{dt}\langle \eta, \varphi(\cdot)\xi\rangle\right)(t_k^n) \in -P(t_k^n, \varphi(t_k^n))(\eta, \xi) + \langle \eta, p(t_k^n)\xi\rangle$$

holds for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$; h^n and the step function $\overline{p}^n$ defined by $\overline{p}^n(t) = p(t_k^n)$ for $t \in (t_{k-1}^n, t_k^n]$ satisfy

$$\lim_{n\to\infty}\|h^n - h\|_{1,\eta\xi}^{(n)} = 0 = \lim_{n\to\infty}\|\overline{p}^n - p\|_{1,\eta\xi}^{(n)}$$

for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$; and the sesq($\mathbb{D} \underline{\otimes} \mathbb{E}$)-valued map Ψ^n on $[s, T(n)]$ defined by

$$\Psi^n(t)(\eta, \xi) = \left(\frac{d}{dt}\langle\eta, \varphi(\cdot)\xi\rangle\right)(t_k^n) \qquad \text{for} \quad t \in (t_{k-1}^n, t_k^n]$$

satisfies $\lim_{n\to\infty}\|\Psi^n - \Psi\|_{1,\eta\xi}^{(n)} = 0$, where Ψ is the sesq($\mathbb{D} \underline{\otimes} \mathbb{E}$)-valued map on $[s, T(n)]$ defined by $(\eta, \xi) \mapsto (d/dt)\langle\eta, \varphi(\cdot)\xi\rangle$, $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$. Define $x_k^n = \varphi(t_k^n)$. Then, (vii) is satisfied and one has

$$\frac{\langle\eta, (x_k^n - x_{k-1}^n)\xi\rangle}{t_k^n - t_{k-1}^n} \in -P(t_k^n, x_k^n)(\eta, \xi) + \langle\eta, p_k^n\xi\rangle$$

for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, where $\{p_k^n\}_{k=1}^{N(n)}$ are defined by

$$\langle\eta, p_k^n\xi\rangle = \langle\eta, \bar{p}_k^n\xi\rangle\left(\frac{d}{dt}\langle\eta, \varphi(\cdot)\xi\rangle\right)(t_k^n) + \frac{\langle\eta, (\varphi(t_k^n) - \varphi(t_{k-1}^n))\xi\rangle}{t_k^n - t_{k-1}^n}$$

Define p^n by $p^n(t) = p_k^n$ for $t \in (t_{k-1}^n, t_k^n]$. It only remains now to show that

$$\lim_{n\to\infty}\|p^n - p\|_{1,\eta\xi}^{(n)} = 0 \tag{$*$}$$

for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$. Note that we already have

$$\lim_{n\to\infty}\|\bar{p}^n - p\|_{1,\eta\xi}^{(n)} = 0$$

for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$. Define the sesq($\mathbb{D} \underline{\otimes} \mathbb{E}$)-valued map ω^n on $[s, T(n)]$ by

$$\omega^n(t)(\eta, \xi) = \frac{\langle\eta, (\varphi(t_k^n) - \varphi(t_{k-1}^n))\xi\rangle}{t_k^n - t_{k-1}^n} \qquad \text{for} \quad t \in (t_{k-1}^n, t_k^n], \quad \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$$

Then

$$\begin{aligned}
&\int_{[s,T(n)]} dt\left|\omega^n(t)(\eta, \xi) - \left(\frac{d}{dt}\langle\eta, \varphi^n(\cdot)\xi\rangle\right)(t)\right| \\
&= \sum_{k=1}^{N(n)}(t_k^n - t_{k-1}^n)\left|\frac{\langle\eta, (\varphi(t_k^n) - \varphi(t_{k-1}^n))\xi\rangle}{t_k^n - t_{k-1}^n} - \left(\frac{d}{dt}\langle\eta, \varphi(\cdot)\xi\rangle\right)(t_k^n)\right| \\
&\leq \sum_{k=1}^{N(n)}\int_{t_{k-1}^n}^{t_k^n} dt\left|\frac{d}{dt}\langle\eta, \varphi(t)\xi\rangle - \left(\frac{d}{dt}\langle\eta, \varphi(\cdot)\xi\rangle\right)(t_k^n)\right| \\
&= \int_{[s,T(n)]} dt\,|\Psi(t)(\eta, \xi) - \Psi^n(t)(\eta, \xi)| \\
&= \|\Psi - \Psi^n\|_{1,\eta\xi}^{(n)}
\end{aligned}$$

which tends to zero as $n \to \infty$, by the choice of partitions. It follows that (*) holds. This concludes the proof. ■

Remark. There is the following corollary of Theorem 8.1.

Corollary 8.2. Let the hypotheses of Theorem 5.1 or Theorem 5.6 hold and φ be a solution of Problem $(3.2.1)_X$ on $[0, \bar{T}]$. If φ_∞ denotes the limit solution described in (vii) of the theorems, then $\varphi = \varphi_\infty$.

Proof. Let $\epsilon > 0$ be given. As φ and φ_∞ are in $C([0, \bar{T}], \mathbb{D} \underline{\otimes} \mathbb{E})$ with $\varphi(0) = x_0 = \varphi_\infty(0)$, there is some $s \in (0, \bar{T})$ such that $\|\varphi(\tau) - \varphi_\infty(\tau)\|_{\eta,\xi} \le \epsilon$ for $0 \le \tau \le s$. Choose T such that $0 < s < T < \bar{T}$.

Using the notation of Section 7, we may write $\varphi_\infty(t) = U(t, 0)x_0$, and both Theorem 5.1 and Theorem 5.6 hold with the initial value $U(s, 0)x_0$. Hence, $U(t, 0)x_0$ may be constructed as the limit of solutions of approximating schemes on $[s, \bar{T}]$, satisfying (i)–(vii) of Theorem 5.1. By Theorem 8.1, φ is also the limit of solutions of approximating schemes. Comparing the two approximating schemes whose solutions converge to φ_∞ and φ gives an inequality as in Proposition 6.1, which upon taking limits gives

$$\|\varphi(t) - \varphi_\infty(t)\|_{\eta,\xi} \le \|\varphi(s) - \varphi_\infty(s)\|_{\eta,\xi}$$

for all $t \in [s, T]$ and arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$. By the choice of s, this gives

$$\|\varphi(t) - \varphi_\infty(t)\|_{\eta,\xi} \le \epsilon$$

for all $t \in [0, T]$ and arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$. It follows that $\varphi = \varphi_\infty$, as $\epsilon > 0$ was arbitrary. This concludes the proof. ■

APPENDIX

The following is a restatement of Proposition 3.4 of Ekhaguere (1995), in which the phrase *maximal monotone* had been inadvertently omitted.

Proposition 3.4 (of Ekhaguere, 1995). Let $\mathcal{P}$ be a regular, maximal monotone multifunction from $\tilde{\mathcal{A}}$ into $2^{\mathrm{sesq}((\mathbb{D}\underline{\otimes}\mathbb{E})^{(2)})}$ and $\alpha, \beta \in L^\infty_{\mathbb{Y},\mathrm{loc}}(\mathbb{R}_+)$. Then:

(i) The multifunction $\mathcal{P}_{\alpha\beta\alpha\beta}$ has convex and τ_w-closed values in $2^{(\tilde{\mathcal{A}}\underline{\otimes}\tilde{\mathcal{A}})_{\alpha\beta}}$.

(ii)(a) If $x \in \tilde{\mathcal{A}}$, $a \in (\tilde{\mathcal{A}} \underline{\otimes} \tilde{\mathcal{A}})_{\alpha\beta}$, $\{x_\delta: \delta \in \Delta\}$ is a net that τ_s-converges to x, $a_\delta \in \mathcal{P}_{\alpha\beta\alpha\beta}(x_\delta)$, and the net $\{a_\delta: \delta \in \Delta\}$ τ_w-converges to a, then $a \in \mathcal{P}_{\alpha\beta\alpha\beta}(x)$.

(ii)(b) If $\mathcal{P}$ if of the form $\mathcal{P} = P \otimes 1$, $x \in \tilde{\mathcal{A}}$, $a \in \tilde{\mathcal{A}}_{\alpha\beta} \otimes 1$, $\{x_\delta: \delta \in \Delta\}$ is a net that τ_w-converges to x, $a_\delta \in \mathcal{P}_{\alpha\beta\alpha\beta}(x_\delta)$, and the net $\{a_\delta: \delta \in \Delta\}$ τ_w-converges to a, then $a \in \mathcal{P}_{\alpha\beta\alpha\beta}(x)$. ■

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